HARMONIC ANALYSIS ASSOCIATED WITH A GENERALIZED BESSEL-STRUVE OPERATOR ON THE REAL LINE

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Abstract. In this paper we consider a generalized Bessel-Struve operator \( l_{\alpha,n} \) on the real line, which generalizes the Bessel-Struve operator \( l_{\alpha} \), we define the generalized Bessel-Struve intertwining operator which turn out to be transmutation operator between \( l_{\alpha,n} \) and the second derivative operator \( \frac{d^2}{dx^2} \). We build the generalized Weyl integral transform and we establish an inversion theorem of the generalized Weyl integral transform. We exploit the generalized Bessel-Struve intertwining operator and the generalized Weyl integral transform, firstly to develop a new harmonic analysis on the real line corresponding to \( l_{\alpha,n} \), and secondly to introduce and study the generalized Sonine integral transform \( S_{\alpha,\beta}^{n,m} \). We prove that \( S_{\alpha,\beta}^{n,m} \) is a transmutation operator from \( l_{\alpha,n} \) to \( l_{\beta,n} \). As a side result we prove Paley-Wiener theorem for the generalized Bessel-Struve transform associated with the generalized Bessel-Struve operator.

I. INTRODUCTION

In this paper we consider the generalized Bessel-Struve operator \( l_{\alpha,n} \), \( \alpha > -\frac{1}{2} \), defined on \( \mathbb{R} \) by

\[
(1) \quad l_{\alpha,n} u(x) = \frac{d^2 u}{dx^2}(x) + \frac{2\alpha + 1}{x} \frac{du}{dx}(x) - \frac{4n(\alpha + n)}{x^2} u(x) - \frac{(2\alpha + 4n + 1)}{x} D(u)(0)
\]

where \( D = x^{2n} \frac{d}{dx} \circ x^{-2n} \) and \( n = 0, 1, ... \). For \( n = 0 \), we regain the Bessel-Struve operator

\[
(2) \quad l_{\alpha} u(x) = \frac{d^2 u}{dx^2}(x) + \frac{2\alpha + 1}{x} \left[ \frac{du}{dx}(x) - \frac{du}{dx}(0) \right].
\]

Through this paper, we provide a new harmonic analysis on the real line corresponding to the generalized Bessel-Struve operator \( l_{\alpha,n} \).

The outline of the content of this paper is as follows.

Section 2 is dedicated to some properties and results concerning the Bessel-Struve transform.

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In section 3, we construct a pair of transmutation operators $X_{\alpha,n}$ and $W_{\alpha,n}$, afterwards we exploit these transmutation operators to build a new harmonic analysis on the real line corresponding to operator $l_{\alpha,n}$.

II. Preliminaries

Throughout this paper assume $\alpha > \beta > -\frac{1}{2}$. We denote by

- $E(\mathbb{R})$ the space of $C^\infty$ functions on $\mathbb{R}$, provided with the topology of compact convergence for all derivatives. That is the topology defined by the seminorms
  \[ p_{a,m}(f) = \sup_{x \in [-a,a]} |f^{(k)}(x)|, \quad a > 0, \ m \in \mathbb{N}, \text{ and } 0 \leq k \leq m. \]
- $D_a(\mathbb{R})$, the space of $C^\infty$ functions on $\mathbb{R}$, which are supported in $[-a,a]$, equipped with the topology induced by $E(\mathbb{R})$.
- $D(\mathbb{R}) = \bigcup_{a>0} D_a(\mathbb{R})$, endowed with inductive limit topology.
- $L^p_{\alpha}(\mathbb{R})$ the class of measurable functions $f$ on $\mathbb{R}$ for which $\|f\|_{p,\alpha} < \infty$, where
  \[ \|f\|_{p,\alpha} = \left( \int_{\mathbb{R}} |f(x)|^p |x|^{2\alpha+1} \, dx \right)^{\frac{1}{p}}, \quad \text{if } p < \infty, \]
  and $\|f\|_{\infty,\alpha} = \|f\|_\infty = \text{ess sup}_{x \geq 0} |f(x)|$.
- $\frac{d}{dx^2} = \frac{1}{2x} \frac{d}{dx}$, where $\frac{d}{dx}$ is the first derivative operator.

In this section we recall some facts about harmonic analysis related to the Bessel-Struve operator $l_\alpha$. We cite here, as briefly as possible, only some properties. For more details we refer to [2, 3].

For $\lambda \in \mathbb{C}$, the differential equation:

\begin{equation}
\begin{cases}
  l_\alpha u(x) = \lambda^2 u(x) \\
  u(0) = 1, \quad u'(0) = \frac{\lambda \Gamma(\alpha+1)}{\sqrt{\pi} \Gamma(\alpha+\frac{3}{2})}
\end{cases}
\end{equation}

possesses a unique solution denoted $\Phi_\alpha(\lambda x)$. This eigenfunction, called the Bessel-Struve kernel, is given by:

\[ \Phi_\alpha(\lambda x) = j_\alpha(i\lambda x) - ih_\alpha(i\lambda x), \quad x \in \mathbb{R}. \]

$j_\alpha$ and $h_\alpha$ are respectively the normalized Bessel and Struve functions of index $\alpha$. These kernels are given as follows

\[ j_\alpha(z) = \Gamma(\alpha+1) \sum_{k=0}^{+\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{k! \Gamma(k+\alpha+1)} \]

and

\[ h_\alpha(z) = \Gamma(\alpha+1) \sum_{k=0}^{+\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k+1}}{\Gamma(k+\frac{3}{2}) \Gamma(k+\alpha+\frac{3}{2})}. \]
The kernel $\Phi_\alpha$ possesses the following integral representation:

\[
\Phi_\alpha(\lambda x) = \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})} \int_0^1 (1 - t^2)^{\alpha - \frac{1}{2}} e^{\lambda xt} dt, \quad \forall x \in \mathbb{R}, \quad \forall \lambda \in \mathbb{C}.
\]

The Bessel-Struve intertwining operator on $\mathbb{R}$ denoted $\mathcal{X}_\alpha$ introduced by L. Kamoun and M. Sifi in [3], is defined by:

\[
\mathcal{X}_\alpha(f)(x) = a_\alpha \int_0^1 (1 - t^2)^{\alpha - 1} f(xt) dt, \quad f \in E(\mathbb{R}), \quad x \in \mathbb{R},
\]

where

\[
a_\alpha = \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})}.
\]

The Bessel-Struve kernel $\Phi_\alpha$ is related to the exponential function by

\[
\forall x \in \mathbb{R}, \quad \forall \alpha \in \mathbb{C}, \quad \Phi_\alpha(\lambda x) = \mathcal{X}_\lambda(e^{\lambda x})(x).
\]

$\mathcal{X}_\alpha$ is a transmutation operator from $l_\alpha$ into $\frac{dx^2}{x^\alpha}$ and verifies

\[
l_\alpha \circ \mathcal{X}_\alpha = \mathcal{X}_\alpha \circ \frac{d^2}{dx^2}.
\]

**Theorem 1.** The operator $\mathcal{X}_\alpha$, $\alpha > -\frac{1}{2}$ is topological isomorphism from $E(\mathbb{R})$ onto itself. The inverse operator $\mathcal{X}_\alpha^{-1}$ is given for all $f \in E(\mathbb{R})$ by

(i) if $\alpha = r + k$, $k \in \mathbb{N}$, $-\frac{1}{2} < r < \frac{1}{2}$

\[
\mathcal{X}_\alpha^{-1}(f)(x) = \frac{2\sqrt{\pi}}{\Gamma(\alpha + 1)\Gamma(\frac{1}{2} - r)} x^k \int_0^x (x^2 - t^2)^{-\frac{1}{2} - r} f(t) |t|^{2\alpha + 1} dt
\]

(ii) if $\alpha = \frac{1}{2} + k$, $k \in \mathbb{N}$

\[
\mathcal{X}_\alpha^{-1}(f)(x) = \frac{2k!}{(2k + 1)!} x^k \int_0^x (x^2 - t^2)^{\frac{1}{2} - k} f(t) x^{2k + 1} dt
\]

**Definition 1.** The Bessel-Struve transform is defined on $L_\alpha^1(\mathbb{R})$ by

\[
\forall \lambda \in \mathbb{R}, \quad \mathcal{F}_\alpha(f)(\lambda) = \int_\mathbb{R} f(x)\Phi_\alpha(-i\lambda x)|x|^{2\alpha + 1} dx.
\]

**Proposition 1.** If $f \in L_\alpha^1(\mathbb{R})$ then $\|\mathcal{F}_\alpha^B,S(f)\|_\infty \leq |f|_{1,\alpha}$.

**Theorem 2.** (Paley-Wiener) Let $a > 0$ and $f$ a function in $\mathcal{D}_\alpha(\mathbb{R})$ then $\mathcal{F}_\alpha^B,S$ can be extended to an analytic function on $\mathbb{C}$ that we denote again $\mathcal{F}_\alpha^B,S(f)$ verifying

\[
\forall k \in \mathbb{N}^*, \quad |\mathcal{F}_\alpha^B,S(f)(z)| \leq C e^{a|z|}.
\]

**Definition 2.** For $f \in L_\alpha^1(\mathbb{R})$ with bounded support, the integral transform $W_\alpha$, given by

\[
W_\alpha(f)(x) = \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})} \int_{|x|}^{+\infty} (y^2 - x^2)^{\alpha - \frac{1}{2}} y f(sgn(x)y) dy, \quad x \in \mathbb{R}\{0\}
\]

is called Weyl integral transform associated with Bessel-Struve operator.
Proposition 2.  
(i) \( W_\alpha \) is a bounded operator from \( L^1_\alpha(\mathbb{R}) \) to \( L^1(\mathbb{R}) \), where \( L^1(\mathbb{R}) \) is the space of lebesgue-integrable functions.

(ii) Let \( f \) be a function in \( E(\mathbb{R}) \) and \( g \) a function in \( L_\alpha(\mathbb{R}) \) with bounded support, the operators \( X_\alpha \) and \( W_\alpha \) are related by the following relation

\[
\int_\mathbb{R} X_\alpha(f)(x)g(x)|x|^{2\alpha+1}dx = \int_\mathbb{R} f(x)W_\alpha(g)(x)dx.
\]

(iii) \( \forall f \in L^1_\alpha(\mathbb{R}) \), \( \mathcal{F}_{R,S} = \mathcal{F} \circ W_\alpha(f) \) where \( \mathcal{F} \) is the classical Fourier transform defined on \( L^1(\mathbb{R}) \) by

\[
\mathcal{F}(g)(\lambda) = \int_\mathbb{R} g(x)e^{-i\lambda x}dx.
\]

We designate by \( K_0 \) the space of functions \( f \) infinitely differentiable on \( \mathbb{R}^* \) with bounded support verifying for all \( n \in \mathbb{N} \),

\[
\lim_{y \to 0^-} y^n f^{(n)}(y) \quad \text{and} \quad \lim_{y \to 0^+} y^n f^{(n)}(y)
\]

exist.

Definition 3. We define the operator \( V_\alpha \) on \( K_0 \) as follows

- If \( \alpha = k + \frac{1}{2}, \ k \in \mathbb{N} \)

\[
V_\alpha(f)(x) = (-1)^{k+1} \frac{2^{2k+1}k!}{(2k+1)!} \left( \frac{d}{dx^2} \right)^{k+1}(f(x)), \ x \in \mathbb{R}^*.
\]

- If \( \alpha = k + r, \ k \in \mathbb{N}, \ \frac{1}{2} < r < \frac{1}{2} \) and \( f \in K_0 \)

\[
V_\alpha(f)(x) = \frac{(-1)^{k+1}2\sqrt{\pi}}{\Gamma(\alpha+1)\Gamma\left(\frac{1}{2} - r\right)} \left[ \int_{|x|}^{\infty} \left( y^2 - x^2 \right)^{-r-\frac{1}{2}} \left( \frac{d}{dy^2} \right)^{k+1} f(sgn(x)y)dy \right], \ x \in \mathbb{R}^*.
\]

Proposition 3. Let \( f \in K_0 \) and \( g \in E(\mathbb{R}) \),

- the operators \( V_\alpha \) and \( X_\alpha^{-1} \) are related by the following relation

\[
\int_\mathbb{R} V_\alpha(f)(x)g(x)|x|^{2\alpha+1}dx = \int_\mathbb{R} f(x)X_\alpha^{-1}(g)(x)dx.
\]

- \( V_\alpha \) and \( W_\alpha \) are related by the following relation

\[
V_\alpha(W_\alpha(f)) = W_\alpha(V_\alpha(f)) = f.
\]

Definition 4. Let \( f \) be a continuous function on \( \mathbb{R} \). We define the Sonine integral transform as in [4] by, for all \( x \in \mathbb{R} \)

\[
S_{\alpha,\beta}(f)(x) = c(\alpha, \beta) \int_0^1 (1 - r^2)^{\alpha-\beta-1} f(rx)r^{\beta+1}dr,
\]

where

\[
c(\alpha, \beta) = \frac{2\Gamma(\alpha + 1)}{\Gamma(\beta + 1)\Gamma(\alpha - \beta)}.
\]
Proposition 4. (i) The classical Sonine integral formula may be formulated as follows

\[ \Phi_\alpha(\lambda x) = c(\alpha, \beta) \int_0^1 (1 - t^2)^{\alpha - \beta - 1} \Phi_\beta(\lambda tx) t^{2\beta + 1} dt. \]

(ii) The Sonine integral transform verifies

\[ S_{\alpha,\beta}(\Phi_\beta(\lambda))(x) = \Phi_\alpha(\lambda x), \quad x \in \mathbb{R}. \]

(iii) For \( f \) a function of class \( C^2 \) on \( \mathbb{R} \), \( S_{\alpha,\beta}(\Phi_\beta(\lambda))(x) \) is a function of class \( C^2 \) on \( \mathbb{R} \) and

\[ \forall x \in \mathbb{R}, \ l_\alpha(S_{\alpha,\beta}(\Phi_\beta(\lambda)))(x) = S_{\alpha,\beta}(l_\beta(f))(x). \]

(iv) The Sonine integral transform is a topological isomorphism from \( E(\mathbb{R}) \) onto itself. Furthermore, it verifies

\[ S_{\alpha,\beta} = \mathcal{X}_\alpha \circ \mathcal{X}_\beta^{-1}. \]

(v) The inverse operator is

\[ S_{\alpha,\beta}^{-1} = \mathcal{X}_\beta \circ \mathcal{X}_\alpha^{-1}. \]

Definition 5. For \( f \) continuous function on \( \mathbb{R} \), with compact support, we define the Dual Sonine transform denoted \( tS_{\alpha,\beta} \) by

\[ tS_{\alpha,\beta}(f)(x) = c(\alpha, \beta) \int_{|x|}^\infty (y^2 - x^2)^{\alpha - \beta - 1} y f(\text{sgn}(x)y) dy, \quad x \in \mathbb{R}^*. \]

Theorem 3. The dual Sonine transform verifies the following relations for all \( f \in D(\mathbb{R}) \) and \( g \in E(\mathbb{R}) \), we have

(i) \[ \int_{\mathbb{R}} S_{\alpha,\beta}(g)(x) f(x) |x|^{2\alpha + 1} dx = \int_{\mathbb{R}} tS_{\alpha,\beta}(f)(x) g(x) |x|^{2\beta + 1} dx. \]

(ii) \[ tS_{\alpha,\beta}(f) = V_\beta(W_\alpha(f)). \]

(iii) \[ \mathcal{F}_{B,S}^\beta(f) = \mathcal{F}_{B,S}^\alpha \circ tS_{\alpha,\beta}(f). \]

III. Harmonic analysis associated with \( l_{\alpha,n} \)

Throughout this section assume \( \alpha > \beta > -\frac{1}{2} \) and \( n = 0, 1, 2, \ldots \). We denoted by

- \( \mathcal{M}_n \) the map defined by \( \mathcal{M}_n f(x) = x^{2n} f(x) \).
- \( E_n(\mathbb{R}) \) (resp \( D_n(\mathbb{R}) \)) stand for the subspace of \( E(\mathbb{R}) \) (resp. \( D(\mathbb{R}) \)) consisting of functions \( f \) such that
  \[ f(0) = \ldots = f^{(2n-1)}(0) = 0. \]
- \( D_{a,n}(\mathbb{R}) = D_a(\mathbb{R}) \cap E_n(\mathbb{R}) \) where \( a > 0 \).
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\( L^p_{a,n}(\mathbb{R}) \) the class of measurable functions \( f \) on \( \mathbb{R} \) for which
\[
\|f\|_{p,a,n} = \|M_n^{-1}f\|_{p,a+2n} < \infty.
\]

i. Transmutation operators.

For \( \lambda \in \mathbb{C} \) and \( x \in \mathbb{R} \), put
\[
\Psi_{\lambda,a,n}(x) = x^{2n}\Phi_{\alpha+2n}(\lambda x)
\]
where \( \Phi_{\alpha+2n} \) is the Bessel-Struve kernel of index \( \alpha+2n \).

**Lemma 1.** (i) The map \( M_n \) is a topological isomorphism

- from \( E(\mathbb{R}) \) onto \( E_n(\mathbb{R}) \).
- from \( D(\mathbb{R}) \) onto \( D_n(\mathbb{R}) \).

(ii) For all \( f \in E(\mathbb{R}) \)
\[
l_{a,n} \circ M_n(f) = M_n \circ l_{\alpha+2n}(f).
\]

**Proof.** Assertion (i) is easily checked (see [1]).

By (1) and (2) we have for any \( f \in E(\mathbb{R}) \),
\[
l_{a,n}(x^{2n}f)(x) = (x^{2n}f)'' + \frac{2\alpha+1}{x}(x^{2n}f)' - \frac{4n(\alpha+n)}{x^2}(x^{2n}f(x)) - (2\alpha+4n+1)x^{2n-1}f'(0)
\]
\[
= x^{2n}\left(f''(x) - \frac{2\alpha+4n+1}{x}(f'(x) - f'(0))\right)
\]
\[
= x^{2n}l_{\alpha+2n}f(x).
\]
which proves Assertion (ii). \( \blacksquare \)

**Proposition 5.** (i) \( \Psi_{\lambda,a,n} \) satisfies the differential equation
\[
l_{a,n}\Psi_{\lambda,a,n} = \lambda^2\Psi_{\lambda,a,n}.
\]

(ii) \( \Psi_{\lambda,a,n} \) possesses the following integral representation:
\[
\Psi_{\lambda,a,n}(x) = \frac{2\Gamma(\alpha+2n+1)}{\sqrt{\pi}\Gamma(\alpha+2n+\frac{1}{2})}x^{2n}\int_0^1 (1-t^2)^{\alpha+2n-\frac{1}{2}}e^{\lambda xt}dt, \quad \forall x \in \mathbb{R}, \forall \lambda \in \mathbb{C}.
\]

**Proof.**

By (23)
\[
\Psi_{\lambda,a,n} = M_n(\Phi_{\alpha+2n}(\lambda x)),
\]
using (3) and (24) we obtain
\[
l_{a,n}(\Psi_{\lambda,a,n}) = l_{a,n} \circ M_n(\Phi_{\alpha+2n}(\lambda))
\]
\[
= M_n \circ l_{\alpha+2n}(\Phi_{\alpha+2n}(\lambda))
\]
\[
= \lambda^2\Psi_{\lambda,a,n},
\]
which proves (i). Statement (ii) follows from (4) and (23). \( \blacksquare \)
Definition 6. For \( f \in E(\mathbb{R}) \), we define the generalized Bessel-Struve intertwining operator \( X_{\alpha,n} \) by

\[
X_{\alpha,n}(f)(x) = a_{\alpha+2n} \int_0^1 (1 - t^2)^{\alpha+2n-1} f(x t) dt , f \in E(\mathbb{R}), \ x \in \mathbb{R}
\]

where \( a_{\alpha+2n} \) is given by (6).

Remark 1. • For \( n = 0 \), \( X_{\alpha,n} \) reduces to the Bessel-Struve intertwining operator.
  • It is easily checked that (25)
    \[
    X_{\alpha,n} = M_n \circ X_{\alpha+2n}.
    \]
  • Due to (7), (23) and (25) we have
    \[
    \Psi_{\lambda,\alpha,n}(x) = X_{\alpha,n}(e^{\lambda}) (x).
    \]

Proposition 6. \( X_{\alpha,n} \) is a transmutation operator from \( l_{\alpha,n} \) into \( \frac{d^2}{dx^2} \) and verifies

\[
l_{\alpha,n} \circ X_{\alpha,n} = X_{\alpha,n} \circ \frac{d^2}{dx^2}.
\]

Proof. It follows from (8), (25) and Lemma 1 (ii) that

\[
l_{\alpha,n} \circ X_{\alpha,n} = l_{\alpha,n} \circ M_n X_{\alpha+2n}
= M_n \circ l_{\alpha+2n} X_{\alpha+2n}
= M_n X_{\alpha+2n} \circ \frac{d^2}{dx^2}
= X_{\alpha,n} \circ \frac{d^2}{dx^2}.
\]

Theorem 4. The operator \( \mathcal{X}_{\alpha,n} \) is an isomorphism from \( E(\mathbb{R}) \) onto \( E_n(\mathbb{R}) \). The inverse operator \( \mathcal{X}_{\alpha,n}^{-1} \) is given for all \( f \in E_n(\mathbb{R}) \) by

(i) if \( \alpha = r + k, \ k \in \mathbb{N}, \ -\frac{1}{2} < r < \frac{1}{2} \)

\[
\mathcal{X}_{\alpha,n}^{-1} f(x) = \frac{2^{2k} \pi}{\Gamma(\alpha + 2n + 1)} \left( \frac{d}{dx} \right)^{k+2n+1} \left[ \int_0^x (x^2 - t^2)^{-\frac{1}{2} - r} f(t) t^{2\alpha + 2n + 1} dt \right].
\]

(ii) if \( \alpha = \frac{1}{2} + k, \ k \in \mathbb{N} \)

\[
\mathcal{X}_{\alpha,n}^{-1} f(x) = \frac{2^{2k+4n+1} (k + 2n)!}{(2k + 4n + 1)!} \left( \frac{d}{dx} \right)^{k+2n+1} (x^{2k+2n+1} f(x)), x \in \mathbb{R}.
\]

Proof. A combination of (25), Lemma 1 and Theorem 1 shows that \( \mathcal{X}_{\alpha,n} \) is an isomorphism from \( E(\mathbb{R}) \) onto \( E_n(\mathbb{R}) \). Let \( \mathcal{X}_{\alpha,n}^{-1} \) the inverse operator of \( \mathcal{X}_{\alpha,n} \), we have

\[
\mathcal{X}_{\alpha,n}^{-1}(f) = (\mathcal{X}_{\alpha,n}(f))^{-1}.
\]

Using (25) we can deduce that

\[
\mathcal{X}_{\alpha,n}^{-1}(f) = (\mathcal{M}_n \mathcal{X}_{\alpha+2n}(f))^{-1}
\]

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By (9) and (10) we obtain the desired result.

ii. The generalized Weyl integral transform.

**Definition 7.** For \( f \in L^1_{\alpha,n}(\mathbb{R}) \) with bounded support, the integral transform \( W_{\alpha,n} \), given by

\[
W_{\alpha,n}(f(x)) = a_{\alpha+2n} \int_{|x|}^{+\infty} (y^2 - x^2)^{\alpha+2n-\frac{1}{2}} y^{1-2n} f(sgn(x)y)dy, \quad x \in \mathbb{R}\setminus\{0\}
\]

is called the generalized Weyl integral transform associated with Bessel-Struve operator.

**Remark 2.**

- By a change of variable, \( W_{\alpha,n}f \) can be written

\[
W_{\alpha,n}f(x) = a_{\alpha+2n} |x|^{2\alpha+2n+1} \int_{1}^{+\infty} (t^2 - 1)^{\alpha+2n-\frac{1}{2}} t^{1-2n} f(tx) dt, \quad x \in \mathbb{R}\setminus\{0\}.
\]

- It is easily checked that

\[
W_{\alpha,n} = W_{\alpha+2n} \circ M_n^{-1}.
\]

**Proposition 7.** \( W_{\alpha,n} \) is a bounded operator from \( L^1_{\alpha,n}(\mathbb{R}) \) to \( L^1(\mathbb{R}) \), where \( L^1(\mathbb{R}) \) is the space of lebesgue-integrable.

**Proof.** Let \( f \in L^1_{\alpha,n}(\mathbb{R}) \), by Proposition 2 (i) we can find a positif constant \( C \) such that

\[
\|W_{\alpha+2n}(M_n^{-1}f)\|_1 \leq C\|M_n^{-1}f\|_{1,\alpha+2n}
\]

\[
\|W_{\alpha,n}(f)\|_1 \leq C\|f\|_{1,\alpha,n}.
\]

By (27) we obtain the desired result. ■

**Proposition 8.** Let \( f \) be a function in \( E(\mathbb{R}) \) and \( g \) a function in \( L^1_{\alpha,n}(\mathbb{R}) \) with bounded support, the operators \( X_{\alpha,n} \) and \( W_{\alpha,n} \) are related by the following relation

\[
\int_{\mathbb{R}} X_{\alpha,n}(f)(x)g(x)|x|^{2\alpha+1}dx = \int_{\mathbb{R}} f(x)W_{\alpha,n}(g)(x)dx.
\]
Proof. Using (25), (27) and Proposition 2 (ii) we obtain

\[ \int_{\mathbb{R}} \mathcal{X}_{\alpha,n}(f(x))g(x)|x|^{2\alpha+1}dx = \int_{\mathbb{R}} \mathcal{M}_n \mathcal{X}_{\alpha+2n}(f(x))g(x)|x|^{2\alpha+1}dx \]
\[ = \int_{\mathbb{R}} x^{2n} \mathcal{X}_{\alpha+2n}(f(x))g(x)|x|^{2\alpha+1}dx \]
\[ = \int_{\mathbb{R}} \mathcal{X}_{\alpha+2n}f(x) \frac{g(x)}{x^{2n}} |x|^{2\alpha+4n+1}dx \]
\[ = \int_{\mathbb{R}} f(x) \mathcal{X}_{\alpha+2n}(g(x))dx \]
\[ = \int_{\mathbb{R}} f(x) \mathcal{M}_n(g(x))dx. \]

\[ \blacksquare \]

Definition 8. We define the operator \( V_{\alpha,n} \) on \( K_0 \) as follows

- If \( \alpha = k + \frac{1}{2}, k \in \mathbb{N} \) and \( f \in K_0 \)
  \[ V_{\alpha,n}f(x) = (-1)^{k+1} \frac{2^{2k+4n+1}(k+2n)!}{(2k+4n+1)!} x^{2n} \left( \frac{d}{dx} \right)^{k+2n+1} f(x), \quad x \in \mathbb{R}^*. \]

- If \( \alpha = k + r, k \in \mathbb{N}, \frac{1}{2} \leq r < \frac{1}{2} \)
  \[ V_{\alpha,n}f(x) = \frac{(-1)^{k+1} 2^{2k+4n+1}}{\Gamma(\alpha+2n+1) \Gamma(\frac{1}{2} - r)} x^{2n} \left( \int_{|x|}^\infty (y^2 - x^2)^{r-\frac{1}{2}} \left( \frac{d}{dy} \right)^{k+2n+1} f(sgn(y)y)dy \right), \quad x \in \mathbb{R}^*. \]

Remark 3. It is easily checked that

\[ (28) \]
\[ V_{\alpha,n} = \mathcal{M}_n \circ V_{\alpha+2n}. \]

Proposition 9. Let \( f \in K_0 \) and \( g \in E_n(\mathbb{R}) \), the operators \( V_{\alpha,n} \) and \( \mathcal{X}_{\alpha,n}^{-1} \) are related by the following relation

\[ \int_{\mathbb{R}} V_{\alpha,n}f(x)g(x)|x|^{2\alpha+1}dx = \int_{\mathbb{R}} f(x) \mathcal{X}_{\alpha,n}^{-1}g(x)dx. \]

Proof. A combination of (14), (26) and (28) shows that

\[ \int_{\mathbb{R}} V_{\alpha,n}(f(x))g(x)|x|^{2\alpha+1}dx = \int_{\mathbb{R}} \mathcal{M}_n V_{\alpha+2n}(f(x))g(x)|x|^{2\alpha+1}dx \]
\[ = \int_{\mathbb{R}} x^{2n} V_{\alpha+2n}(f(x))g(x)|x|^{2\alpha+1}dx \]
\[ = \int_{\mathbb{R}} V_{\alpha+2n}(f(x)) \frac{g(x)}{x^{2n}} |x|^{2\alpha+4n+1}dx \]
\[ = \int_{\mathbb{R}} f(x) \mathcal{X}_{\alpha+2n}^{-1}(\frac{g(x)}{x^{2n}})dx \]
\[ = \int_{\mathbb{R}} f(x) \mathcal{X}_{\alpha,n}^{-1}(g(x))dx. \]

\[ \blacksquare \]
Theorem 5. Let \( f \in K_0 \), \( V_{\alpha,n} \) and \( W_{\alpha,n} \) are related by the following relation
\[
V_{\alpha,n}(W_{\alpha,n}(f)) = W_{\alpha,n}(V_{\alpha,n}(f)) = f.
\]

Proof. The result follows directly from Proposition 3.(15), (27) and (28). ■

iii. The generalized Sonine integral transform.

Definition 9. Let \( f \in E_m(\mathbb{R}) \). We define the generalized Sonine integral transform by, for all \( x \in \mathbb{R} \)
\[
S_{\alpha,\beta}^{n,m}(f)(x) = c(\alpha + 2n, \beta + 2m)x^{2(n-m)}\int_0^1 (1 - t^2)^{\alpha-\beta+2(n-m)-1}f(rt)\gamma^2 + 2m + 1 \, dr,
\]
where \( \alpha > \beta > -\frac{1}{2} \) and \( m, n \) two non-negative integers such that \( n \geq m \). For \( n = m = 0 \), \( S_{\alpha,\beta}^{n,m} \) reduces to the classical Sonine integral transform \( S_{\alpha,\beta} \).

Remark 4. Due to (16) and (29)
\[
S_{\alpha,\beta}^{n,m} = M_n \circ S_{\alpha+2n,\beta+2m} \circ M_m^{-1}.
\]

In the next Proposition, we establish an analogue of Sonine formula

Proposition 10. We have the following relation
\[
\Psi_{\lambda,\alpha,n}(x) = c(\alpha + 2n, \beta + 2m)x^{2(n-m)}\int_0^1 (1 - t^2)^{\alpha-\beta+2(n-m)-1}\Psi_{\lambda,\beta,m}(tx)\gamma^2 + 2m + 1 \, dt.
\]

Proof. A combination of (18) and (23) leads to the desired result. ■

Remark 5. The following relation yields from relation (31)
\[
S_{\alpha,\beta}^{n,m}(\Psi_{\lambda,\alpha,n}(.))(x) = \Psi_{\lambda,\alpha,n}(x).
\]

Theorem 6. The generalized Sonine integral transform \( S_{\alpha,\beta}^{n,m}(f) \) is an isomorphism from \( E_m(\mathbb{R}) \) onto \( E_n(\mathbb{R}) \) satisfying the intertwining relation
\[
l_{\alpha,n}(S_{\alpha,\beta}^{n,m}(f))(x) = S_{\alpha,\beta}^{n,m}(l_{\beta,m}(f))(x).
\]

Proof. An easily combination of (20), (24), (30), Lemma 1.(i) and Proposition 4 (iv) yields \( S_{\alpha,\beta}^{n,m}(f) \) is an isomorphism from \( E_m(\mathbb{R}) \) onto \( E_n(\mathbb{R}) \) and
\[
l_{\alpha,n}(S_{\alpha,\beta}^{n,m}(f))(x) = l_{\alpha,n}M_n \circ S_{\alpha+2n,\beta+2m} \circ M_m^{-1}(f)(x)
= M_n l_{\alpha+2n}(S_{\alpha+2n,\beta+2m}) \circ M_m^{-1}(f)(x)
= M_n S_{\alpha+2n,\beta+2m}l_{\beta+2m} \circ M_m^{-1}(f)(x)
= M_n S_{\alpha+2n,\beta+2m}l_{\beta,m}(f)(x)
= S_{\alpha,\beta}^{n,m}(l_{\beta,m}(f))(x).
\]
Theorem 7. The generalized Sonine transform is a topological isomorphism from $E_m(\mathbb{R})$ onto $E_n(\mathbb{R})$. Furthermore, it verifies

$$S^{n,m}_{\alpha,\beta} = X_{\alpha,n} \circ X^{-1}_{\beta,m}$$

the inverse operator is

$$(S^{n,m}_{\alpha,\beta})^{-1} = X_{\beta,m} \circ X^{-1}_{\alpha,n}.$$  

Proof. It follows from (25), (30), Lemma 1.(i) and Proposition 4 ((iv)-(v)) that $S^{n,m}_{\alpha,\beta}$ is a topological isomorphism from $E_m(\mathbb{R})$ onto $E_n(\mathbb{R})$ and

$$S^{n,m}_{\alpha,\beta}(f) = M_n \circ S_{\alpha+2n,\beta+2m} \circ M^{-1}_m(f)$$

$$= M_n X_{\alpha+2n} \circ X^{-1}_{\beta+2m} M^{-1}_m(f)$$

$$= X_{\alpha,n} \circ X^{-1}_{\beta,m}(f).$$

For the inverse operator it is easily checked that

$$(S^{n,m}_{\alpha,\beta})^{-1} = X_{\beta,m} \circ X^{-1}_{\alpha,n}.$$  

Definition 10. For $f \in D_n(\mathbb{R})$ we define the dual generalized Sonine transform denoted $^{t}S_{\alpha,\beta}$ by

$$^{t}S^{n,m}_{\alpha,\beta}(f)(x) = c(\alpha + 2n, \beta + 2m)x^{2m} \int_{|x|}^{\infty} (y^2 - x^2)^{\alpha - \beta + 2(n-m) - 1} y^{1 - 2n} f(sgn(x)y) dy,$$

where $x \in \mathbb{R}^*$.  

Remark 6. Due to (32) and Definition 5 we have

$$^{t}S^{n,m}_{\alpha,\beta} = M_m^{1} \circ S_{\alpha+2n,\beta+2m} \circ M^{-1}_n.$$  

Proposition 11. The dual generalized Sonine transform verifies the following relation for all $f \in D_n(\mathbb{R})$ and $g \in E_m(\mathbb{R})$,

$$\int g(x) f(x)|x|^{2n+1} dx = \int ^{t}S^{n,m}_{\alpha,\beta}(f)(x) g(x)|x|^{2\beta+1} dx.$$  

Proof. A combination of (30), (33) and Theorem 3.(i) we get

$$\int g(x) f(x)|x|^{2n+1} dx = \int M_n \circ S_{\alpha+2n,\beta+2m} \circ M^{-1}_m(g(x)f(x))|x|^{2n+1} dx$$

$$= \int S_{\alpha+2n,\beta+2m} \circ M^{-1}_m(g(x)f(x))|x|^{2(\alpha+2n)+1} dx$$

$$= \int M^{-1}_m(g(x))S_{\alpha+2n,\beta+2m} \circ M^{-1}_n(f(x))|x|^{2(\beta+2m)+1} dx$$

$$= \int M_m(g(x))^{t}S_{\alpha+2n,\beta+2m} \circ M^{-1}_n(f(x))|x|^{2\beta+1} dx$$

$$= \int g(x)^{t}S^{n,m}_{\alpha,\beta}(f)(x)|x|^{2\beta+1} dx.$$
Theorem 8. For all $f \in D_n(\mathbb{R})$, we have
\[ t^{S_{\alpha,\beta}^n}(f) = V_{\beta,m}(W_{\alpha,n}(f)). \]

Proof. By (27), (28), (33) and Theorem 3 (ii), we get
\[ t^{S_{\alpha,\beta}^n}(f) = M_m t^{S_{\alpha+2n,\beta+2m}^n}(f) = M_m V_{\beta+2m}(W_{\alpha+2n}^{-1})(f) = V_{\beta,m}(W_{\alpha,n}(f)). \]

iv. Generalized Bessel-Struve transform.

Definition 11. The Generalized Bessel-Struve transform is defined on $L_{1,\alpha,n}(\mathbb{R})$ by
\[ \forall \lambda \in \mathbb{R}, \quad F_{B,S}^{\alpha,n}(f)(\lambda) = \int_{\mathbb{R}} f(x) \Psi_{-i\lambda,\alpha,n}(x) |x|^{2\alpha+1} dx. \]

Remark 7. It follows from (11), (23) and Definition 11 that
\[ F_{B,S}^{\alpha,n} = F_{B,S}^{\alpha+2n} \circ M_{n}^{-1}, \] where $F_{B,S}^{\alpha+2n}$ is the Bessel-Struve transform of order $\alpha + 2n$ given by (11).

Proposition 12. If $f \in L_{1,\alpha,n}(\mathbb{R})$ then
(i) $\|F_{B,S}^{\alpha,n}(f)\|_{\infty} \leq \|f\|_{1,\alpha,n}$.
(ii) $F_{B,S}^{\alpha,n} = F \circ W_{\alpha,n}$.

Proof. (i) By Remark 7 and Proposition 1, we have for all $f \in L_{1,\alpha,n}(\mathbb{R})$
\[ \|F_{B,S}^{\alpha,n}(f)\|_{\infty} = \|F_{B,S}^{\alpha+2n}(M_{n}^{-1}f)\|_{\infty} \leq \|M_{n}^{-1}f\|_{1,\alpha+2n} = \|f\|_{1,\alpha,n}. \]

(ii) From (27), Remark 7 and Proposition 2.(iii), we have for all $f \in L_{1,\alpha,n}(\mathbb{R})$
\[ F_{B,S}^{\alpha,n}(f) = F_{B,S}^{\alpha+2n} \circ M_{n}^{-1}(f) = F \circ W_{\alpha+2n}(M_{n}^{-1}(f)) = F \circ W_{\alpha,n}(f). \]

Proposition 13. For all $f \in D_{n}(\mathbb{R})$, we have the following decomposition
\[ F_{B,S}^{\alpha,n}(f) = F_{B,S}^{\beta,m} \circ t^{S_{\alpha,\beta}^n}(f). \]
**Proof.** It follows from (33), Remark 7 and Theorem 3.(iii) that
\[
\mathcal{F}_{B,S}^{\alpha,n}(f) = \mathcal{F}_{B,S}^{\beta+2m} \circ M^{-1}_n(f)
= \mathcal{F}_{B,S}^{\beta+2m} \circ t_{\alpha+2n,\beta+2m} \circ M^{-1}_n(f)
= \mathcal{F}_{B,S}^{\beta+2m} \circ M^{-1}_m \circ t_{\alpha+2n,\beta+2m} \circ M^{-1}_n(f)
= \mathcal{F}_{B,S}^{\beta,m} \circ t_{\alpha,\beta}^n(f).
\]

**Theorem 9.** (Paley-Wiener) Let \( a > 0 \) and \( f \) a function in \( D_{a,n}(\mathbb{R}) \) then \( \mathcal{F}_{B,S}^{\alpha,n} \) can be extended to an analytic function on \( \mathbb{C} \) that we denote again \( \mathcal{F}_{B,S}^{\alpha,n}(f) \) verifying
\[
\forall k \in \mathbb{N}^*, \quad |\mathcal{F}_{B,S}^{\alpha,n}(f)(z)| \leq C e^{a|z|}.
\]

**Proof.** The result follows directly from Remark 7, Lemma 1(i) and Theorem 2.

**REFERENCES**


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