A New Class of Contra Continuous Functions in Topological Spaces

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ABSTRACT: In this paper, we introduce and investigate the notion of contra πgr-continuous, almost contra πgr-continuous functions and discussed the relationship with other contra continuous functions and obtained their characteristics.

Keywords: Contra πgr-continuous, almost contra πgr-continuous, πgr-locally indiscrete, T_{πgr}-space.
AMS Subject Classification: 54C08, 54C10

I. INTRODUCTION


In this paper, the notion of contra πgr-continuity which is a stronger form of contra πg-continuity and their characterizations are introduced and investigated. Further, the notion of almost contra πgr-continuity is introduced and its properties are discussed.

II. PRELIMINARIES

In the present paper, the spaces X and Y always mean topological spaces (X,τ) and (Y,σ) respectively. For a subset A of a space, cl(A) and int(A) represent the closure of A and interior of A respectively.

Definition: 2.1
A subset A of X is said to be regular open [13] if A=int(cl(A)) and its complement is regular closed.

The finite union of regular open set is a regular open set[21] and its complement is a regular closed set. The union of all regular open sets contained in A is called rint(A)[regular interior of A] and the intersection of regular closed sets containing A is called rcl(A)[regular closure of A]

Definition: 2.2
A subset A of X is called
1. gr -closed[12,14] if rcl(A) ⊂ U whenever A⊂U and U is open.
2. πgr-closed[9] if rcl(A) ⊂ U whenever A⊂U and U is π-open.

Definition: 2.3
A function f: (X,τ)→(Y,σ) is called πgr-continuous[9] if f^{−1}(V) is πgr-closed in X for every closed set V in Y.

Definition: 2.4
A function f: (X,τ)→(Y,σ) is called
(i) Contra continuous[2] if f^{−1}(V) is closed in X for each open set V of Y.
(ii) Contra πg-continuous[5] if f^{−1}(V) is πg-closed in X for each open set V of Y.
(iii) Contra πgr-continuous[8] if f^{−1}(V) is πgr-closed in X for each open set V of Y.
(iv) Contra πgb-continuous[18] if f^{−1}(V) is πgb-closed in X for each open set V of Y.
(v) Contra π^g-continuous[6] if f^{−1}(V) is π^g-closed in X for each open set V of Y.
(vi) Contra gr-continuous[12] if f^{−1}(V) is gr-closed in X for each open set V of Y.
(vii) RC-continuous[5] if f^{−1}(V) is regular closed in X for each open set V of Y.
(viii) An R-map [5] if f^{−1}(V) is regular closed in X for each regular closed set V of Y.
(ix) Perfectly continuous[4] if f^{−1}(V) is clopen in X for each open set V of Y.
(x) rc-preserving [5] if f(U) is regular closed in Y for each regular closed set U of X.
(xi) A function \( f : X \to Y \) is called regular set connected [5] if \( f^{-1}(V) \) is clopen in \( X \) for every \( V \) in \( Y \).

(xii) Contra R-map [5] if \( f^{-1}(V) \) is regular closed in \( X \) for each regular open set \( V \) of \( Y \).

(xiii) Almost continuous [15] if \( f^{-1}(V) \) is closed in \( X \) for every regular closed set \( V \) of \( Y \).

**Definition : 2.5**

A space \((X,\tau)\) is called

(i) \(\pi\)-gr-T_{12} space [8] if every \(\pi\)-gr-closed set is regular closed.

(ii) locally indiscrete [20] if every open subset of \( X \) is closed.

(iii) Weakly Hausdorff [17] if each element of \( X \) is an intersection of regular closed sets.

(iv) Ultra Hausdorff space [19], if for every pair of distinct point \( x \) and \( y \) in \( X \), there exist clopen sets \( U \) and \( V \) in \( X \) containing \( x \) and \( y \) respectively.

(v) Hyper connected [20] if every open set is dense.

**Definition : 2.6**

A collection \( \{ A_i : i \in \Lambda \} \) of open sets in a topological space \( X \) is called open cover [16] of a subset \( B \) of \( X \) if \( B \subset \bigcup \{ A_i : i \in \Lambda \} \).

**Definition : 2.7**

A collection \( \{ A_i : i \in \Lambda \} \) of \(\pi\)-gr-open sets in a topological space \( X \) is called \(\pi\)-gr-open cover [10] of a subset \( B \) of \( X \) if \( B \subset \bigcup \{ A_i : i \in \Lambda \} \).

**Definition : 2.8**

A space \( X \) is called \(\pi\)-gr-connected [10] provided that \( X \) is not the union of two disjoint non-empty \(\pi\)-gr-open sets.

**Definition : 2.9** [5]

Let \( S \) be a closed subset of \( X \). The set \( \bigcap \{ U \in \tau / S \subset U \} \) is called the kernel of \( S \) and is denoted by \( \text{Ker}(S) \).

**III. CONTRA \(\pi\)-GR-CONTINUOUS FUNCTION.**

**Definition : 3.1**

A function \( f : (X,\tau) \to (Y,\sigma) \) is called Contra \(\pi\)-gr-continuous if \( f^{-1}(V) \) is \(\pi\)-gr-closed in \( (X,\tau) \) for each open set \( V \) of \( (Y,\sigma) \).

**Definition : 3.2**

A space \((X,\tau)\) is called

(i) \(\pi\)-gr-locally indiscrete if every \(\pi\)-gr-open set is closed.

(ii) \(T_{\pi\text{-gr}}\)-space if every \(\pi\)-gr-closed is gr-closed.

**Result : 3.3**

Contra Continuous and contra \(\pi\)-gr-continuous are independent concepts.

**Example : 3.4**

a) Let \( X = \{a,b,c,d\} = Y, \tau = \{\varnothing, X, \{a\}, \{b\}, \{a,b,c\}, \{a,b,d\} \} \) \(\sigma = \{\varnothing, Y, \{c\}\} \). Let \( f : X \to Y \) be an identity map. Here the inverse image of the element \( c \) in the open set of \( Y \) is closed in \( X \) but not \(\pi\)-gr-closed in \( X \). Hence \( f \) is contra continuous and not contra \(\pi\)-gr-continuous.

b) Let \( X = \{a,b,c,d\} = Y, \tau = \{\varnothing, X, \{a\}, \{b\}, \{a,b,c\}, \{a,b,d\} \} \), \(\sigma = \{\varnothing, Y, \{d\}, \{a,d\}\} \). Let \( f : X \to Y \) be an identity map. Here the inverse image of the elements in the open set of \( Y \) are \(\pi\)-gr-closed in \( X \) but not closed in \( X \). Hence \( f \) is contra \(\pi\)-gr-continuous and not contra continuous. Hence contra continuity and contra \(\pi\)-gr-continuity are independent concepts.

**Theorem : 3.5**

Every RC-continuous function is contra \(\pi\)-gr-continuous but not conversely.

**Proof:** Straight Forward.

**Example : 3.6**

Let \( X = \{a,b,c,d\} = Y, \tau = \{\varnothing, X, \{a\}, \{c,d\}, \{a,c,d\}\} \), \(\sigma = \{\varnothing, Y, \{a\}, \{a,b\}\} \). Let \( f : X \to Y \) be defined by \( f(a) = b, f(b) = a, f(c) = c, f(d) = d \). The inverse image of the element in the open set of \( Y \) is \(\pi\)-gr-closed in \( X \) but not regular closed in \( X \). Hence \( f \) is contra \(\pi\)-gr-continuous and not RC-continuous.

**Theorem : 3.7**

Every Contra gr-continuous function is contra \(\pi\)-gr-continuous but not conversely.

**Proof:** Follows from the definition.

**Example : 3.8**

Let \( X = \{a,b,c,d\}, \tau = \{\varnothing, X, \{c\}, \{d\}, \{c,d\}, \{b,d\}, \{a,c,d\}, \{b,c,d\}\} \), \(\sigma = \{\varnothing, Y, \{a\}, \{a,d\}\} \). The inverse image of the element \( \{a,d\} \) in the open set of \( Y \) is \(\pi\)-gr-closed in \( X \) but not gr-closed. Hence \( f \) is contra \(\pi\)-gr-continuous and not contra gr-continuous.

**Theorem : 3.9**

Every contra \(\pi\)-gr-continuous function is contra \(\pi\)-continuous, contra \(\pi^{*}\)-continuous, contra \(\pi\)-gr-continuous and contra \(\pi\)-grb-continuous but not conversely.
**Proof:** Straight Forward.

**Example: 3.10**

a.Let X = \{a,b,c,d\}, \tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,b,c,d\}\}, \sigma = \{\emptyset, Y, \{b\}\}. Here the inverse image of the element \{b\} in the open set \(Y, \sigma\) is \(\pi g\)-closed in X, but not \(\pi gr\)-closed in X. Hence \(f\) is contra \(\pi g\)-continuous and not contra \(\pi gr\)-continuous.

b.Let X = \{a,b,c,d\}, \tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,b,c,d\}\}, \sigma = \{\emptyset, Y, \{b\}\}. Here the inverse image of the element \{b\} in the open set \(Y, \sigma\) is \(\pi g\)-closed in X, but not \(\pi gr\)-closed in X. Hence \(f\) is contra \(\pi g\)-continuous and not contra \(\pi gr\)-continuous.

c.Let X = \{a,b,c,d\} - Y, \tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,b,c,d\}\}, \sigma = \{\emptyset, Y, \{a\}, \{a,b,c,d\}\}. Let \(f: X \rightarrow Y\) be an identity map. The inverse image of the element \{a\} in the open set \(Y, \sigma\) is \(\pi gb\)-closed but not \(\pi gr\)-closed. Hence \(f\) is contra \(\pi gb\)-continuous and not contra \(\pi gr\)-continuous.

d.Let X = \{a,b,c,d\} - Y, \tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,b,c,d\}\}, \sigma = \{\emptyset, Y, \{c\}, \{d\}\}. Let \(f: X \rightarrow Y\) be an identity map. The inverse image of all the elements in Y are \(\pi gb\)-closed but not \(\pi gr\)-closed. Hence \(f\) is contra \(\pi gb\)-continuous and not contra \(\pi gr\)-continuous.

**Remark: 3.11**

The above relations are summarized in the following diagram.

![Diagram](https://www.ijres.org/46/46.png)

**Theorem: 3.12**

Suppose \(\pi gro(X, \tau)\) is closed under arbitrary unions. Then the following are equivalent for a function \(f\) : \((X, \tau) \rightarrow (Y, \sigma)\):

1. \(f\) is contra \(\pi gr\)-continuous.
2. For every closed subset \(F\) of Y, \(f^{-1}(F)\) \(\in\) \(\pi gro(X, \tau)\)
3. For each \(x \in X\) and each \(F \in C(Y, f(x))\), there exists a set \(U \in \pi gro(X, x)\) such that \(f(U) \subseteq F\).

**Proof:**

(1) \(\Leftrightarrow\) (2): Let \(f\) is contra \(\pi gr\)-continuous. Then \(f^{-1}(V)\) is \(\pi gr\)-closed in X for every open set \(V\) of \(Y\). (i.e) \(f^{-1}(F)\) is \(\pi gr\)-open in X for every closed set \(F\) of \(Y\). Hence \(f^{-1}(F)\) \(\in\) \(\pi gro(X, \tau)\).

(2) \(\Rightarrow\) (3): For every closed subset \(F\) of Y, \(f^{-1}(F)\) \(\in\) \(\pi gro(X, \tau)\). Then for each \(x \in X\) and each \(F \in C(Y, f(x))\), there exists a set \(U \in \pi gro(X, x)\) such that \(f(U) \subseteq F\).

(3) \(\Rightarrow\) (2): For each \(x \in X\), \(f \in C(Y, f(x))\), there exists a set \(U \in \pi gro(X, x)\) such that \(f(U) \subseteq F\). Let \(F\) be a closed set of \(Y\) and \(x \in f^{-1}(F)\). Then \(f(x) \in F\). Hence \(f^{-1}(F)\) \(\subseteq\) \(\pi gr\)-open.

**Theorem: 3.13**

If \(f: X \rightarrow Y\) is contra \(\pi gr\)-continuous and \(U\) is open in X. Then \(f(U) : (U, \tau) \rightarrow (Y, \sigma)\) is contra \(\pi gr\)-continuous.

**Proof:**

Let \(V\) be any closed set in \((Y, \sigma)\). Since \(f(X, \tau) \rightarrow (Y, \sigma)\) is contra \(\pi gr\)-continuous, \(f^{-1}(V)\) is \(\pi gr\)-open in X. Hence \(f(U)\) is \(\pi gr\)-open in Y.

**Theorem: 3.14**

If a function \(f\) : \((X, \tau) \rightarrow (Y, \sigma)\) is \(\pi gr\)-continuous and the space \((X, \tau)\) is \(\pi gr\)-locally indiscrete , then \(f\) is contra continuous.

**Proof:**

Let \(V\) be a open set in \((Y, \sigma)\). Since \(f\) is \(\pi gr\)-continuous, \(f^{-1}(V)\) is open in X. Hence \(f(U)\) is \(\pi gr\)-open in Y.

**Theorem: 3.15**

If a function \(f: X \rightarrow Y\) is contra \(\pi gr\)-continuous, \(X\) is a \(\pi gr\) -T\(_{1/2}\) space, then \(f\) is \(RC\)-continuous.

**Proof:**

Let \(V\) be open in \(Y\). Since \(f\) is contra \(\pi gr\)-continuous, \(f^{-1}(V)\) is \(\pi gr\)-closed in X. Hence \(f(U)\) is \(\pi gr\)-closed in X. Therefore \(f(U)\) is \(RC\)-continuous.

**Theorem: 3.16**

If a function \(f: (X, \tau) \rightarrow (Y, \sigma)\) is contra \(\pi gr\)-continuous, \(RC\)-preserving surjection and if \(X\) is a \(\pi gr\) -T\(_{1/2}\) space, then \(Y\) is locally indiscrete.
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Proof: Let V be open in Y. Since f is contra \( \pi \)gr-continuous, \( f^1(V) \) is \( \pi \)gr-closed in X. Since X is a \( \pi \)gr-\( T_{1/2} \) space, \( f^1(V) \) is regular closed in X. Since f is \( \pi \)gr-preserving surjection, \( f(f^1(V)) = V \) is regular closed in Y. Thus \( cl(V) = cl(int(V)) \subset cl(cl(int(V))) \subset V \). Hence V is closed in Y. Therefore, Y is locally indiscrete.

Theorem 3.17
If a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is contra \( \pi \)gr-continuous and X is a \( \pi \)gr-space, then \( f((X, \tau)) - (Y, \sigma) \) is contra \( \pi \)gr-continuous.

Proof: Let V be an open set in Y. Since f is contra \( \pi \)gr-continuous, \( f^1(V) \) is \( \pi \)gr-closed in X. Since X is a \( T_{1/2} \) space, \( f^1(V) \) is gr-closed in X. Thus for every open set V of Y, \( f^1(V) \) is gr-closed. Hence f is contra \( \pi \)gr-continuous.

Theorem 3.18
Suppose \( \pi \)GRO(X, \tau) is closed under arbitrary unions. Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a function and \( \{U_i : i \in I = 1, 2, \ldots \} \) be a cover of X such that \( U_i \not\in \pi \)GRC(X, \tau) and regular open for each \( i \in I \). If \( f(U_i : (U_i, \tau(U_i)) \rightarrow (Y, \sigma) \) is contra \( \pi \)gr-continuous for each \( i \in I \), then f is contra \( \pi \)gr-continuous.

Proof: Suppose that \( F \) is any closed set of Y. We have \( f^1(F) = \bigcup \{ f^1(U_i) : i \in I \} \) since \( f(U_i) \) is contra \( \pi \)gr-continuous for each \( i \in I \). Hence \( f^1(F) \) is \( \pi \)GRO(X).

Theorem 3.19
Suppose \( \pi \)GRO(X, \tau) is closed under arbitrary unions. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is contra \( \pi \)gr-continuous if Y is regular, then f is \( \pi \)g-continuous.

Proof: Let x be an arbitrary point of X and V be an open set of Y containing f(x). The regularity of Y implies that there exists an open set \( W \in Y \) containing f(x) such that \( f(W) \not\subset V \). Since f is contra \( \pi \)gr-continuous, then there exists \( U \in \pi \)GRO(X, \tau) such that \( f(U) \not\subset f(W) \). Then \( f(U) \not\subset V \). Hence f is \( \pi \)gr-continuous.

Theorem 3.20
Suppose that \( \pi \)GR(X) is closed under arbitrary intersections. Then the following are equivalent for a function f: X → Y:

1) f is contra \( \pi \)gr-continuous.
2) The inverse image of every closed set of Y is \( \pi \)gr-open.
3) For each \( x \in X \) and each closed set \( B \) in Y with \( f(x) \in B \), there exists a \( \pi \)gr-open set \( A \) in X such that \( x \in A \) and \( f(A) \in B \).
4) \((\pi \)gr-cl(A)) \subset Ker f(A)\) for every subset \( A \) of X.
5) \((\pi \)gr-cl(f^1(B)) \subset f^1(Ker(B)))\) for every subset \( B \) of Y.

Proof:
(1) ⇒ (2) and (2) ⇒ (1) are obviously true.
(1) ⇒ (3): Let \( x \in X \) and \( B \) be a closed set in Y with \( f(x) \in B \). By (1), it follows that \( f^1(Y - B) = X - f^1(B) \) is \( \pi \)gr-closed and \( f^1(B) \) is \( \pi \)gr-open.
Take \( A = f^1(B) \). We obtain that \( x \in A \) and \( f(A) \subset B \).
(3) ⇒ (2): Let B be a closed set in Y with \( x \in B \). Since \( f(x) \in B \), by (3), there exists a \( \pi \)gr-open set A in X having x such that \( f(A) \subset B \). It follows that \( x \in A \cap f^1(B) \). Hence \( f^1(B) \) is \( \pi \)gr-open.
(2) ⇒ (1): Obvious.
(2) ⇒ (4): Let \( A \) be any subset of X. Let \( y \not\in Ker f(A) \). Then there exists a closed set \( F \) containing y such that \( f(A) \not\subset F \). Hence, we have \( A \cap f^1(F) = \emptyset \). Thus \( f(\pi \)gr-cl(A)) \subset F = \emptyset \) and \( y \not\in f(\pi \)gr-cl(A)) \) and hence \( f(\pi \)gr-cl(A)) \subset Ker f(A) \).
(4) ⇒ (5): Let B be any subset of Y. By (4), \( f(\pi \)gr-cl(f^1(B)) \subset Ker B \) and \( \pi \)gr-cl(f^1(B)) \subset f^1(Ker B) \).
(5) ⇒ (1): Let B be any open set of Y. By (5), \( \pi \)gr-cl(f^1(B)) \subset f^1(Ker B) = f^1(B) \).

IV. ALMOST CONTRA \( \pi \)GR-CONTINUOUS FUNCTIONS.

Definition 4.1
A function \( f : X \rightarrow Y \) is said to be almost contra \( \pi \)gr-continuous if \( f^1(V) \) is closed in X for each regular open set V of Y.

Definition 4.2
A function \( f : X \rightarrow Y \) is said to be almost contra \( \pi \)gr-continuous if \( f^1(V) \) is \( \pi \)gr-closed in X for each regular open set V of Y.

Definition 4.3
A topological space X is said to be \( \pi \)gr-\( T_1 \) space if for any pair of distinct points \( x \) and \( y \), there exists a \( \pi \)gr-open sets \( G \) and \( H \) such that \( x \in G \), \( y \not\in G \) and \( x \not\in H \), \( y \in H \).

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Definition: 4.4
A topological space X is said to be πgr-T2-space if for any pair of distinct points x and y, there exists disjoint πgr-open sets G and H such that x ∉ G and y ∉ H.

Definition: 4.5
A topological space X is said to be πgr-Normal if each pair of disjoint closed sets can be separated by disjoint πgr-open sets.

Definition: 4.32
A function f : X → Y is called Weakly πgr-continuous if for each x ∈ X and each open set V of Y containing f(x), there exists U ∈ πgrO(X, x) such that f(U) ⊂ cl(V).

Definition: 4.7
A space X is said to be 1. πgr-compact if every πgr-open cover of X has a finite sub-cover.
2. Nearly compact if every regular open cover has a finite subcover.
3. Nearly lindelof if every regular open cover of X has a countable subcover.
4. S-lindelof if every cover of X by regular closed sets has a countable subcover.
5. S-closed if every regular closed cover of X has a finite subcover.

Definition: 4.8
A space X is said to be 1. πgr- Lindelof if every πgr-open cover of X has a countable subcover.
2. Mildly πgr-compact if every πgr-clopen cover of X has a finite subcover.
3. Mildly πgr-lindelof if every πgr-clopen cover of X has a countable subcover.
4. Countably πgr-compact if every countable cover of X by πgr-open sets has a finite subcover.

Theorem: 4.9
Suppose πgr-open set of X is closed under arbitrary unions. The following statements are equivalent for a function f : X → Y.
1) f is almost contra πgr- continuous.
2) f∗(F) ∈ πGR(Y) for every F ∈ RC(Y).
3) For each x ∈ X and each regular closed set F in Y containing f(x), there exists a πgr-open set U in X containing x such that f(U) ⊂ F.
4) For each x ∈ X and each regular open set V in Y not containing f(x), there exists a πgr-closed set K in X not containing x such that f∗(V) ⊂ K.
5) f∗(int(cl(G)) ⊂ πGRC(X, τ) for every closed subset F of Y.
6) f∗(cl(int(F))) ∈ πGRO(X, τ) for every closed subset F of Y.

Proof:
1)⇒(2): Let F ∈ RC(Y, σ). Then Y − f(F) ∈ πRO(Y, σ). Since f is almost contra πgr-continuous, f∗(Y − F) = X − f∗(F) ∈ πGR(X). Hence f∗(F) ∈ πGR(X).
2)⇒(1): Let F ∈ RO(Y, σ). Then Y = V ∪ RO(Y, σ). Since for each F ∈ RC(Y, σ),
f∗(Y − V) = X − f∗(V) ∈ πGRO(X)
f∗(V) ∈ πGR(X)
⇒ f is almost contra πgr-continuous.
3)⇒(2): Let F ∈ RC(Y, σ) and x ∈ f∗(F). From (3), there exists a πgr-open set U in X containing x such that U ⊂ f∗(F). We have f∗(F) = U ∪ {x ∈ f∗(F)}. Thus f∗(F) is πgr-open.
4)⇒(3): Let X be a regular open set in Y not containing f(x). Then Y − V is a regular closed set containing f(x).
By (3), there exists a πgr-open set U in X containing x such that f(U) ⊂ Y − V. Hence U ⊂ πGRC(X, τ) − f∗(V).
5)⇒(4): Take K ⊂ X − U. We obtain a πgr-closed set K in X not containing x such that f∗(V) ⊂ K.
6)⇒(5): Let f be a regular closed set in Y containing f(x). Then Y − F is a regular open set in Y containing f(x).
By (4), there exists a πgr-closed set K in X not containing x such that f∗(Y − F) ⊂ K, f∗(F) ⊂ K. Hence X − K ⊂ f∗(F). Hence f(X − K) ⊂ F. Take U = X − K, f(U) ⊂ F. Then U is a πgr-open set in X containing x such that f(U) ⊂ F.

(1)⇒(5): Let G be an open subset of Y. Since int(cl(G)) is regular open, then by (1),
f∗(int(cl(G)) ∈ πGRC(X, τ)
⇒ f is almost contra πgr-continuous.
(5)⇒(1): Let V ∈ RO(Y, σ). Then V is open in X. By (5), f∗(int(cl(V)) ∈ πGRC(X, τ)
⇒ f∗(V) ∈ πGR(X, τ)
⇒ f is almost contra πgr-continuous.
Theorem 4.10
Every contra πgr-continuous function is almost contra πgr-continuous but not conversely.
Proof: Straight forward.

Example 4.11
Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset, X, \{a\}, \{c,d\}, \{a,c,d\}\}$, πgr-closed set=$\{\emptyset, X \setminus \{b\}$ \{a\}, \{b\}, \{c\}, \{d\}\}. Let $Y = \{a,b,c,d\}$, $\sigma = \{\emptyset, Y, \{a\}, \{b\}\}$. Let $f$ be an identity map. The inverse image of open set in $Y$ is not πgr-closed in $X$. But the inverse image of regular open set in $Y$ is πgr-closed in $X$. Hence $f$ is almost contra πgr-continuous and not contra πgr-continuous.

Theorem 4.12
Every regular set connected function is almost contra πgr-continuous but not conversely.

Example 4.13
Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$, $\tau' = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$. Let $Y = \{a,b,c\}$, $\sigma = \{\emptyset, Y, \{a\}, \{b\}\}$. Let $f$ be an identity map. The inverse image of open set $\{a\}$ is not clopen in $X$. But the inverse image of open set in $Y$ is πgr-closed in $Y$. Hence $f$ is almost contra πgr-continuous and not regular set connected.

Theorem 4.14
Let $f:X \to Y$, $g:Y \to Z$ be two functions. Then the following properties hold.
a) If $f$ is almost contra πgr-continuous and $g$ is regular set connected, then $gof : X \to Z$ is almost contra πgr-continuous and almost πgr-continuous.
b) If $f$ is almost contra πgr-continuous and $g$ is perfectly continuous, then $gof : X \to Z$ is πgr-continuous and contra πgr-continuous.
c) If $f$ is contra πgr-continuous and $g$ is regular set connected, then $gof : X \to Z$ is πgr-continuous and almost πgr-continuous.

Proof:
a) Let $V \in \text{RO}(Z)$. Since $g$ is regular set connected, $g^{-1}(V)$ is clopen in $Y$. Since $f$ is almost contra πgr-continuous, $f^{-1}(g^{-1}(V)) = (gof)^{-1}(V)$ is πgr-open and πgr-closed. Therefore, $(gof)$ is almost contra πgr-continuous and almost πgr-continuous.
b) Let $V$ be open in $Z$. Since $g$ is perfectly continuous, $g^{-1}(V)$ is clopen in $Y$. Since $f$ is almost contra πgr-continuous, $f^{-1}(g^{-1}(V)) = (gof)^{-1}(V)$ is πgr-open and πgr-closed. Hence $gof$ is contra πgr-continuous and πgr-continuous.
c) Let $V \in \text{RO}(Z)$. Since $g$ is regular set connected, $g^{-1}(V)$ is clopen in $Y$. Since $f$ is a contra πgr-continuous, $f^{-1}(g^{-1}(V)) = (gof)^{-1}(V)$ is πgr-closed in $X$. Therefore, $(gof)$ is πgr-continuous and almost πgr-continuous.

Theorem 4.15
If $f:X \to Y$ is an almost contra πgr-continuous, injection and $Y$ is weakly hausdorff, then $X$ is πgr-$T_1$.

Proof: Suppose $Y$ is weakly hausdorff. For any distinct points $x$ and $y$ in $X$, there exists $V$ and $W$ regular closed sets in $Y$ such that $f(x) \notin V, f(y) \notin W$ and $f(x) \notin W$. Since $f$ is almost contra πgr-continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are πgr-open subsets of $X$ such that $x \in f^{-1}(V), y \notin f^{-1}(V), y \notin f^{-1}(W)$ and $x \notin f^{-1}(W)$. This shows that $X$ is πgr-$T_1$.

Corollary 4.16
If $f:X \to Y$ is a contra πgr-continuous injection and $Y$ is weakly hausdorff, then $X$ is πgr-$T_1$.

Proof: Since every contra πgr-continuous function is almost contra πgr-continuous, the result of this corollary follows by using the above theorem.

Theorem 4.17
If $f:X \to Y$ is an almost contra πgr-continuous injective function from space $X$ to a ultra Hausdorff space $Y$, then $X$ is πgr-$T_2$.

Proof: Let $x$ and $y$ be any two distinct points in $X$. Since $f$ is injective, $f(x) \neq f(y)$ and $Y$ is Ultra Hausdorff space, there exists disjoint clopen sets $U$ and $V$ of $Y$ containing $f(x)$ and $f(y)$ respectively. Then $x \in f^{-1}(U), y \notin f^{-1}(V), y \notin f^{-1}(W)$ and $x \notin f^{-1}(V)$. This shows that $X$ is πgr-$T_2$.

Theorem 4.18
If $f:X \to Y$ is an almost contra πgr-continuous injection and $Y$ is Ultra Normal. Then $X$ is πgr-normal.

Proof: Let $G$ and $H$ be disjoint closed subsets of $X$. Since $f$ is closed and injective, $f(E)$ and $f(F)$ are disjoint closed sets in $Y$. Since $Y$ is Ultra Normal, there exists disjoint clopen sets $U$ and $V$ in $Y$ such that $f(G) \subseteq U$ and $f(H) \subseteq V$. Hence $G \cap f^{-1}(U), H \cap f^{-1}(V)$. Since $f$ is an almost contra πgr-continuous injective function, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint πgr-open sets in $X$. Therefore, $X$ is πgr-$T_2$.

Theorem 4.20
If $f:X \to Y$ is an almost contra πgr-continuous surjection and $X$ is πgr-connected space, then $Y$ is connected.
Proof: Let \( f: X \to Y \) be an almost contra \( \pi gr \)-continuous surjection and \( X \) is \( \pi gr \)-connected space. Suppose \( Y \) is not connected space, then there exists disjoint open sets \( U \) and \( V \) such that \( Y = U \cup V \). Therefore, \( U \) and \( V \) are clopen in \( Y \). Since \( f \) is almost contra \( \pi gr \)-continuous, \( f'(U) \) and \( f'(V) \) are \( \pi gr \)-open sets in \( X \). Moreover, \( f'(U) \) and \( f'(V) \) are non-empty disjoint \( \pi gr \)-open sets and \( X = f'(U) \cup f'(V) \). This is a contradiction to the fact that \( X \) is \( \pi gr \)-connected space. Therefore, \( Y \) is connected.

**Theorem 4.2.1**

If \( X \) is \( \pi gr \)-Ultra connected and \( f: X \to Y \) is an almost contra \( \pi gr \)-continuous surjective, then \( Y \) is hyper connected.

**Proof:** Let \( X \) be a \( \pi gr \)-Ultra connected and \( f: X \to Y \) is an almost contra \( \pi gr \)-continuous surjection. Suppose \( Y \) is not hyper connected. Then there exists an open set \( V \) such that \( V \) is not dense in \( Y \). Therefore, there exists non-empty regular open subsets \( B_1 = \text{int}(\text{cl}(V)) \) and \( B_2 = Y - \text{cl}(V) \) in \( Y \). Since \( f \) is an almost contra \( \pi gr \)-continuous surjection, \( f'(B_1) \) and \( f'(B_2) \) are disjoint \( \pi gr \)-closed sets in \( X \). This is a contradiction to the fact that \( X \) is \( \pi gr \)-ultra connected. Therefore, \( Y \) is hyper connected.

**Theorem 4.2.2**

If a function \( f: X \to Y \) is an almost contra \( \pi gr \)-continuous, then \( f \) is weakly \( \pi gr \)-continuous function.

**Proof:** Let \( x \in X \) and \( V \) be an open set in \( Y \) containing \( f(x) \). Then \( \text{cl}(V) \) is regular closed in \( Y \) containing \( f(x) \). Since \( f \) is an almost contra \( \pi gr \)-continuous function for every regular closed set \( f'(Y) \) is \( \pi gr \)-open in \( X \). Hence \( f'(\text{cl}(V)) \) is \( \pi gr \)-open set in \( X \) containing \( x \). Set \( U = f'(\text{cl}(V)) \), then \( f(U) \subset f'(\text{cl}(V)) \subset \text{cl}(V) \). This shows that \( f \) is weakly \( \pi gr \)-continuous function.

**Theorem 4.2.3**

Let \( f: X \to Y \) be an almost contra \( \pi gr \)-continuous surjection. Then the following properties hold:

1. If \( X \) is \( \pi gr \)-compact, then \( Y \) is \( S \)-closed.
2. If \( X \) is countably \( \pi gr \)-closed, then \( Y \) is countably \( S \)-closed.
3. If \( X \) is \( \pi gr \)-lindelöf, then \( Y \) is \( S \)-lindelöf.

**Proof:**

1) Let \( \{ V_\alpha : \alpha \in I \} \) be any regular closed cover of \( Y \). Since \( f \) is almost contra \( \pi gr \)-continuous, \( \{ f^{-1}(V_\alpha) : \alpha \in I \} \) is \( \pi gr \)-open cover of \( X \). Since \( X \) is \( \pi gr \)-compact, there exists a finite subset \( I_0 \) of \( I \) such that \( X = \cup \{ f^{-1}(V_\alpha) : \alpha \in I_0 \} \). Since \( f \) is surjective, \( Y = \cup \{ V_\alpha : \alpha \in I_0 \} \) is finite subcover for \( Y \). Therefore, \( Y \) is countably \( S \)-closed.

2) Let \( \{ V_\alpha : \alpha \in I \} \) be any countable regular closed cover of \( Y \). Since \( f \) is almost contra \( \pi gr \)-continuous, \( \{ f^{-1}(V_\alpha) : \alpha \in I \} \) is \( \pi gr \)-open cover of \( X \). Since \( X \) is \( \pi gr \)-lindelöf, there exists a finite subset \( I_0 \) of \( I \) such that \( X = \cup \{ f^{-1}(V_\alpha) : \alpha \in I_0 \} \). Since \( f \) is surjective, \( Y = \cup \{ V_\alpha : \alpha \in I_0 \} \) is finite subcover of \( Y \). Therefore, \( Y \) is \( S \)-lindelöf.

3) Let \( \{ V_\alpha : \alpha \in I \} \) be any regular closed cover of \( Y \). Since \( f \) is almost contra \( \pi gr \)-continuous, \( \{ f^{-1}(V_\alpha) : \alpha \in I \} \) is \( \pi gr \)-open cover of \( X \). Since \( X \) is \( \pi gr \)-lindelöf, there exists a countable subset \( I_0 \) of \( I \) such that \( X = \cup \{ f^{-1}(V_\alpha) : \alpha \in I_0 \} \). Since \( f \) is surjective, \( Y = \cup \{ V_\alpha : \alpha \in I_0 \} \) is finite subcover of \( Y \). Therefore, \( Y \) is \( S \)-lindelöf.

**Theorem 4.2.4**

Let \( f: X \to Y \) be an almost contra \( \pi gr \)-continuous and almost continuous surjection. Then the following properties hold.

1) If \( X \) is mildly \( \pi gr \)-closed, then \( Y \) is nearly compact.
2) If \( X \) is mildly countably \( \pi gr \)-compact, then \( Y \) is nearly countably compact.
3) If \( X \) is mildly \( \pi gr \)-lindelöf, then \( Y \) is nearly lindelöf.

**Proof:**

1) Let \( \{ V_\alpha : \alpha \in I \} \) be any open cover of \( Y \). Since \( f \) is almost contra \( \pi gr \)-continuous and almost \( \pi gr \)continuous function, \( \{ f^{-1}(V_\alpha) : \alpha \in I \} \) is \( \pi gr \)-open cover of \( X \). Since \( X \) is mildly \( \pi gr \)-compact, there exists a finite subset \( I_0 \) of \( I \) such that \( X = \cup \{ f^{-1}(V_\alpha) : \alpha \in I_0 \} \). Since \( f \) is surjective, \( Y = \cup \{ V_\alpha : \alpha \in I_0 \} \) is finite subcover for \( Y \). Therefore, \( Y \) is nearly compact.

2) Similar to that of (1).

3) Let \( \{ V_\alpha : \alpha \in I \} \) be any regular open cover of \( Y \). Since \( f \) is almost contra \( \pi gr \)-continuous and almost \( \pi gr \)continuous function, \( \{ f^{-1}(V_\alpha) : \alpha \in I \} \) is \( \pi gr \)-open cover of \( X \). Since \( X \) is mildly \( \pi gr \)-lindelöf, there exists a countable subset \( I_0 \) of \( I \) such that \( X = \cup \{ f^{-1}(V_\alpha) : \alpha \in I_0 \} \). Since \( f \) is surjective, \( Y = \cup \{ V_\alpha : \alpha \in I_0 \} \) is finite subcover for \( Y \). Therefore, \( Y \) is nearly lindelöf.
REFERENCES