Unique Common Fixed Point Theorem for Three Pairs of Weakly Compatible Mappings Satisfying Generalized Contractive Condition of Integral Type

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Abstract: We prove some unique common fixed point result for three pairs of weakly compatible mappings satisfying a generalized contractive condition of Integral type in complete G-metric space. The present theorem is the improvement and extension of Vishal Gupta and Naveen Mani [5] and many other results existing in literature.

Keywords: Fixed point, Complete G- metric space, G-Cauchy sequence, Weakly compatible mapping, Integral Type contractive condition.

I. Introduction


Mustafa in collaboration with Sims [10] introduced a new notation of generalized metric space called G- metric space in 2006. He proved many fixed point results for a self mapping in G- metric space under certain conditions.

Now we give some preliminaries and basic definitions which are used through-out the paper.

Definition 1.1: Let X be a non empty set, and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

$(G_1)$ \hspace{1cm} $G(x, y, z) = 0$ if \hspace{0.5cm} $x = y = z$

$(G_2)$ \hspace{1cm} $0 < G(x, x, y)$ for all $x, y \in X$, with \hspace{0.5cm} $x \neq y$

$(G_3)$ \hspace{1cm} $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with \hspace{0.5cm} $y \neq z$

$(G_4)$ \hspace{1cm} $G(x, y, z) = G(x, z, y) = G(y, z, x)$ \hspace{0.5cm} (Symmetry in all three variables)

$(G_5)$ \hspace{1cm} $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ \hspace{0.5cm} (rectangle inequality)

Then the function G is called a generalized metric space, or more specially a G- metric on X, and the pair $(X, G)$ is called a G- metric space.

Definition 1.2: Let $(X, G)$ be a G- metric space and let $\{x_n\}$ be a sequence of points of $X$, a point $x \in X$ is said to be the limit of the sequence $\{x_n\}$, if $\lim_{m,n \to \infty} G(x, x_n, x_m) = 0$, and we say that the sequence $\{x_n\}$ is G - convergent to $x$ or $\{x_n\}$ G - converges to $x$. 


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Thus, \( x_n \to x \) in a \( G \)-metric space \((X, G)\) if for any \( \varepsilon > 0 \) there exists \( k \in \mathbb{N} \) such that \( G(x, x_n, x_m) < \varepsilon \), for all \( m, n \geq k \).

**Proposition 1.3:** Let \((X, G)\) be a \( G \)-metric space. Then the following are equivalent:

i) \( \{x_n\} \) is \( G \)-convergent to \( x \)

ii) \( G(x_n, x_n, x) \to 0 \) as \( n \to +\infty \)

iii) \( G(x_n, x, x) \to 0 \) as \( n \to +\infty \)

iv) \( G(x_n, x_m, x) \to 0 \) as \( n, m \to +\infty \)

**Proposition 1.4:** Let \((X, G)\) be a \( G \)-metric space. Then for any \( x, y, z, a \in X \) it follows that

i) If \( G(x, y, z) = 0 \) then \( x = y = z \)

ii) \( G(x, y, z) \leq G(x, x, y) + G(x, x, z) \)

iii) \( G(x, y, y) \leq 2G(y, x, x) \)

iv) \( G(x, y, z) \leq G(x, a, z) + G(a, y, z) \)

v) \( G(x, y, z) \leq \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z)) \)

vi) \( G(x, y, z) \leq (G(x, a, a) + G(y, a, a) + G(z, a, a)) \)

**Definition 1.5:** Let \((X, G)\) be a \( G \)-metric space. A sequence \( \{x_n\} \) is called a \( G \)-Cauchy sequence if for any \( \varepsilon > 0 \) there exists \( k \in \mathbb{N} \) such that \( G(x_n, x_m, x_l) < \varepsilon \) for all \( m, n, l \geq k \), that is \( G(x_n, x_m, x_l) \to 0 \) as \( n, m, l \to +\infty \).

**Proposition 1.6:** Let \((X, G)\) be a \( G \)-metric space. Then the following are equivalent:

i) The sequence \( \{x_n\} \) is \( G \)-Cauchy;

ii) For any \( \varepsilon > 0 \) there exists \( k \in \mathbb{N} \) such that \( G(x_n, x_m, x_l) < \varepsilon \) for all \( m, n, l \geq k \)

**Proposition 1.7:** A \( G \)-metric space \((X, G)\) is called \( G \)-complete if every \( G \)-Cauchy sequence is \( G \)-convergent in \((X, G)\).

**Proposition 1.8:** Let \((X, G)\) be a \( G \)-metric space. Then the function \( G(x, y, z) \) is jointly continuous in all three of its variables.

**Definition 1.9:** Let \( f \) and \( g \) be two self – maps on a set \( X \). Maps \( f \) and \( g \) are said to be commuting if \( fgx = gfx \), for all \( x \in X \)

**Definition 1.10:** Let \( f \) and \( g \) be two self – maps defined on a set \( X \), then \( f \) and \( g \) are said to be weakly compatible if they commute at coincidence points. That is if \( fu = gu \) for some \( u \in X \), then \( fgx = gfx \).

The main aim of this paper is to prove a unique common fixed point theorem for three pairs of weakly compatible mappings satisfying integral type contractive condition in a complete \( G \)-metric space.

The result is the extension of the following theorem of Vishal Gupta and Naveen Mani [5].

**II. Theorem**

Let \( S \) and \( T \) be self compatible maps of a complete metric space \((X, d)\) satisfying the following conditions

\[
\psi \int_0^d S(x, y) \, dt \leq \psi \int_0^{d(Tx, Ty)} \phi(t) \, dt - \phi \int_0^{d(Tx, Ty)} \phi(t) \, dt
\]

for each \( x, y \in X \) where \( \psi : [0, +\infty) \to [0, +\infty) \) is a continuous and non decreasing function and \( \phi : [0, +\infty) \to [0, +\infty) \) is a lower semi continuous and non decreasing function such that \( \psi(t) = \phi(t) = 0 \) if and only if \( t = 0 \) also \( \phi : [0, +\infty) \to [0, +\infty) \) is a “Lebesgue-integrable function” which is summable on each
compact subset of $R^+$, nonnegative, and such that for each $\varepsilon > 0$, $
\int_0^\varepsilon \varphi(t)\,dt > 0$. Then $S$ and $T$ have a unique common fixed point.

III. MAIN RESULT

**Theorem 2.1** : Let $(X, G)$ be a complete G-metric space and $L, M, N, P, Q, R : X \to X$ be mappings such that

i) $L(X) \subset P(X)$, $M(X) \subset Q(X)$, $N(X) \subset R(X)$

ii) $\xi \left\{ \begin{array}{l} \frac{G(L_x, M_x, N_x)}{G(P_x, Q_x, R_x)} \int_0^f (t)\,dt \leq \eta \left\{ \begin{array}{l} \frac{G(P_x, Q_x, R_x)}{G(L_x, M_x, N_x)} \int_0^f (t)\,dt \end{array} \right. \\
\end{array} \right.$ \hspace{1cm} \text{-------------------(2.1.1)}$

for all $x, y, z \in X$ where $\xi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function, $\eta : [0, \infty) \to [0, \infty)$ is a lower semi continuous and non-decreasing function such that $\xi(t) = \eta(t) = 0$ if and only if $t = 0$, also $f : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function which is summable on each compact subset of $R^+$, non negative and such that for each $\varepsilon > 0$, $
\int_0^\varepsilon f(t)\,dt > 0$

iii) The pairs $(L, P)$, $(M, Q)$, $(N, R)$ are weakly compatible.

Then $L, M, N, P, Q, R$ have a unique common fixed point in $X$.

**Proof :** Let $x_0$ be an arbitrary point of $X$ and define the sequence $\{x_n\}$ in $X$ such that

$y_m = Lx_m = P_{m+1}$, $y_{n+1} = Mx_{n+1} = Qx_{n+2}$, $y_{n+2} = Nx_{n+2} = Rx_{n+3}$

Consider $\xi \left\{ \begin{array}{l} \frac{G(L_{x_n}, M_{x_{n+1}}, N_{x_{n+2}})}{G(P_{x_{n+2}}, Q_{x_{n+3}}, R_{x_{n+4}})} \int_0^f (t)\,dt \leq \eta \left\{ \begin{array}{l} \frac{G(P_{x_{n+2}}, Q_{x_{n+3}}, R_{x_{n+4}})}{G(L_{x_n}, M_{x_{n+1}}, N_{x_{n+2}})} \int_0^f (t)\,dt \end{array} \right. \\
\end{array} \right.$ \hspace{1cm} \text{-----------------(2.1.2)}$

Since $\xi$ is continuous and has a monotone property.

$\therefore \int_0^\xi f(t)\,dt \leq \int_0^\eta f(t)\,dt$ \hspace{1cm} \text{-----------------(2.1.3)}$

Let us take $\delta_n = \int_0^{G(y_m, y_{n+1}, y_{n+2})} f(t)\,dt$, then it follows that $\delta_n$ is monotone decreasing and lower bounded sequence of numbers.

Therefore there exists $k \geq 0$ such that $\delta_n \to k$ as $n \to \infty$. Suppose that $k > 0$

Taking limit as $n \to \infty$ on both sides of (2.1.2) and using that $\eta$ is lower semi continuous, we get $\xi(k) \leq \xi(k) - \eta(k) < \xi(k)$, which is a contradiction. Hence $k = 0$.

This implies that $\delta_n \to 0$ as $n \to \infty$ i.e. $\int_0^\delta f(t)\,dt \to 0$ as $n \to \infty$. \hspace{1cm} \text{---------}(2.1.4)$
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Now, we prove that \( \{y_n \} \) is a G-Cauchy sequence. On the contrary, suppose it is not a G-Cauchy sequence.

\[ \therefore \text{There exists } \varepsilon > 0 \text{ and subsequences } \{y_{m(i)}\} \text{ and } \{y_{n(i)}\} \text{ such that for each positive integer } i, n(i) \text{ is minimal in the sense that}, \]
\[ G(y_{n(i)}, y_{m(i)}), y_{m(i)}), y_{m(i)}) \leq \varepsilon \text{ and } G(y_{n(i)}, y_{m(i)}), y_{m(i)}) < \varepsilon \]

Now, \( \varepsilon \leq G(y_{n(i)}, y_{m(i)}), y_{m(i)}) \leq G(y_{n(i)}, y_{m(i)}), y_{m(i)}) + G(y_{m(i)}), y_{m(i)}), y_{m(i)}) \quad \text{---------(2.1.5)} \]

Let \( 0 < \alpha = \int_0^\infty f(t) \, dt \leq \int_0^\alpha f(t) \, dt \leq \int_0^\infty G(y_{n(i)}, y_{m(i)}), y_{m(i)}) \]

Taking \( i \to \infty \), and using (2.1.4) , we get \( \lim_{i \to \infty} \int_0^\infty f(t) \, dt = \alpha \quad \text{---------}(2.1.6) \)

Now, using rectangular inequality, we have

\[ G(y_{n(i)}, y_{m(i)}), y_{m(i)}) \leq G(y_{n(i)}, y_{n(i)}), y_{n(i)}) + G(y_{n(i)}, y_{m(i)}), y_{m(i)}) + G(y_{n(i)}, y_{m(i)}), y_{m(i)}) \quad \text{---------(2.1.7)} \]

\[ G(y_{n(i)}, y_{n(i)}), y_{n(i)}) \leq G(y_{n(i)}, y_{n(i)}), y_{n(i)}) + G(y_{n(i)}, y_{n(i)}), y_{n(i)}) + G(y_{n(i)}, y_{n(i)}), y_{n(i)}) \quad \text{---------(2.1.8)} \]

\[ \therefore \int_0^\infty f(t) \, dt \leq \int_0^\infty f(t) \, dt \]

Taking limit as \( i \to \infty \) and using (2.1.4) , (2.1.6) we get

\[ \lim_{i \to \infty} \int_0^\infty f(t) \, dt \leq \alpha \quad \text{---------}(2.1.9) \]

Hence \( \{y_n\} \) is a G-Cauchy sequence. Since \((X, G)\) is a complete G-metric space, there exists a point \( u \in X \) such that \( n \to \infty \), \( y_n = u \)

\[ \lim_{n \to \infty} Lx_n = \lim_{n \to \infty} Px_{n+1} = u \quad \lim_{n \to \infty} Mx_{n+1} = \lim_{n \to \infty} Qx_{n+2} = u \quad \lim_{n \to \infty} Nx_{n+2} = \lim_{n \to \infty} Rx_{n+3} = u \]

As \( Lx_n \to u \) and \( Px_{n+1} \to u \), therefore we can find some \( h \in X \) such that \( Qh = u \).

\[ \xi \left\{ \int_0^\alpha f(t) \, dt \right\} \leq \xi \left\{ \int_0^\alpha f(t) \, dt \right\} - \eta \left\{ \int_0^\alpha f(t) \, dt \right\} \]

\[ \therefore \text{Taking limit as } n \to \infty \text{, we get } \xi \left\{ \int_0^\alpha f(t) \, dt \right\} \leq \xi(0) - \eta(0) \]
Consider, \( \xi \left\{ \int_0^t f(t) \, dt \right\} \geq \xi \left\{ \int_0^t f(t) \, dt \right\} - \eta \left\{ \int_0^t f(t) \, dt \right\} \). On taking limit as \( n \to \infty \), we get \( \xi \left\{ \int_0^t f(t) \, dt \right\} \leq \xi(0) - \eta(0) \)

And therefore, \( Lu = Pu \) i.e. \( u \) is the fixed point of \( L \).

Thus we get \( Nw = Rw = v \) i.e. \( v \) is the coincidence point of \( N \) and \( R \).

Since the pair of maps \( N \) and \( R \) are weakly compatible, we have \( NRw = RNw \) i.e. \( Nu = Ru \)

Now, we prove that \( u \) is a fixed point of \( M \).

Consider, \( \xi \left\{ \int_0^t f(t) \, dt \right\} \geq \xi \left\{ \int_0^t f(t) \, dt \right\} - \eta \left\{ \int_0^t f(t) \, dt \right\} \). On taking limit as \( n \to \infty \), we get \( \xi \left\{ \int_0^t f(t) \, dt \right\} \leq \xi(0) - \eta(0) \)

Hence, \( Mh = u \).

Hence, \( Mh = Qh = u \) i.e. \( h \) is the point of coincidence of \( M \) and \( Q \).

Since the pair of maps \( M \) and \( Q \) are weakly compatible, we write \( MQh = QMh \) i.e. \( Mu = Qu \).

Also, \( Mx_{n+1} \to u \) and \( Qx_{n+1} \to u \), \( \therefore \) we can find some \( v \in X \) such that \( Pv = u \).

\[ \xi \left\{ \int_0^t f(t) \, dt \right\} \leq \xi \left\{ \int_0^t f(t) \, dt \right\} - \eta \left\{ \int_0^t f(t) \, dt \right\} \]

On taking limit as \( n \to \infty \), we get \( \xi \left\{ \int_0^t f(t) \, dt \right\} \leq \xi(0) - \eta(0) \)

Hence, \( Lv = u \). Therefore we can find some \( w \in X \) such that \( Rw = u \).

\[ \xi \left\{ \int_0^t f(t) \, dt \right\} \leq \xi \left\{ \int_0^t f(t) \, dt \right\} - \eta \left\{ \int_0^t f(t) \, dt \right\} \]

On taking limit as \( n \to \infty \), we get \( \xi \left\{ \int_0^t f(t) \, dt \right\} \leq \xi(0) - \eta(0) \)

i.e. \( \xi \left\{ \int_0^t f(t) \, dt \right\} \leq \xi(0) - \eta(0) \)

Thus we get \( Nu = Rw = v \) i.e. \( v \) is the coincidence point of \( N \) and \( R \).

Since the pair of maps \( N \) and \( R \) are weakly compatible, we have \( NRw = RNw \) i.e. \( Nu = Ru \)

Now, we show that \( u \) is the fixed point of \( L \).

Consider, \( \xi \left\{ \int_0^t f(t) \, dt \right\} \geq \xi \left\{ \int_0^t f(t) \, dt \right\} - \eta \left\{ \int_0^t f(t) \, dt \right\} \). On taking limit as \( n \to \infty \), we get \( \xi \left\{ \int_0^t f(t) \, dt \right\} \leq \xi(0) - \eta(0) \)

Hence, \( Mh = u \). Therefore we can find some \( w \in X \) such that \( Rw = u \).

\[ \xi \left\{ \int_0^t f(t) \, dt \right\} \leq \xi \left\{ \int_0^t f(t) \, dt \right\} - \eta \left\{ \int_0^t f(t) \, dt \right\} \]

On taking limit as \( n \to \infty \), we get \( \xi \left\{ \int_0^t f(t) \, dt \right\} \leq \xi(0) - \eta(0) \)

Hence, \( Lv = u \). Therefore we can find some \( w \in X \) such that \( Rw = u \).

\[ \xi \left\{ \int_0^t f(t) \, dt \right\} \leq \xi \left\{ \int_0^t f(t) \, dt \right\} - \eta \left\{ \int_0^t f(t) \, dt \right\} \]

On taking limit as \( n \to \infty \), we get \( \xi \left\{ \int_0^t f(t) \, dt \right\} \leq \xi(0) - \eta(0) \)

Hence, \( Mh = u \). Therefore we can find some \( w \in X \) such that \( Rw = u \).

\[ \xi \left\{ \int_0^t f(t) \, dt \right\} \leq \xi \left\{ \int_0^t f(t) \, dt \right\} - \eta \left\{ \int_0^t f(t) \, dt \right\} \]

On taking limit as \( n \to \infty \), we get \( \xi \left\{ \int_0^t f(t) \, dt \right\} \leq \xi(0) - \eta(0) \)

Hence, \( Mh = u \). Therefore we can find some \( w \in X \) such that \( Rw = u \).

\[ \xi \left\{ \int_0^t f(t) \, dt \right\} \leq \xi \left\{ \int_0^t f(t) \, dt \right\} - \eta \left\{ \int_0^t f(t) \, dt \right\} \]

On taking limit as \( n \to \infty \), we get \( \xi \left\{ \int_0^t f(t) \, dt \right\} \leq \xi(0) - \eta(0) \)

Hence, \( Mh = u \). Therefore we can find some \( w \in X \) such that \( Rw = u \).

\[ \xi \left\{ \int_0^t f(t) \, dt \right\} \leq \xi \left\{ \int_0^t f(t) \, dt \right\} - \eta \left\{ \int_0^t f(t) \, dt \right\} \]

On taking limit as \( n \to \infty \), we get \( \xi \left\{ \int_0^t f(t) \, dt \right\} \leq \xi(0) - \eta(0) \)
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\[ \xi \left\{ \int_0^\infty f(t) \, dt \right\} < \xi \left\{ \int_0^\infty f(t) \, dt \right\}, \text{ which is a contradiction.} \therefore \text{ we get } Mu = u \]

Hence \( Mu = Qu = u \) i.e. \( u \) is fixed point of \( M \) and \( Q \).

At last we prove that \( u \) is fixed point of \( N \).

Consider, \[ \xi \left\{ \int_0^\infty f(t) \, dt \right\} \leq \xi \left\{ \int_0^\infty f(t) \, dt \right\} - \eta \left\{ \int_0^\infty f(t) \, dt \right\} \]

i.e. \[ \xi \left\{ \int_0^\infty f(t) \, dt \right\} < \xi \left\{ \int_0^\infty f(t) \, dt \right\}, \text{ which means } \xi \left\{ \int_0^\infty f(t) \, dt \right\} < \xi \left\{ \int_0^\infty f(t) \, dt \right\} \text{ as } Nu = Ru. \]

Which implies that \( Nu = u \). Hence we get \( Nu = Ru = u \).

i.e. \( u \) is fixed point of \( N \) and \( R \).

Thus \( u \) is the common fixed point of \( L, M, N, P, Q \) and \( R \).

Now, we prove that \( u \) is the unique common fixed point of \( L, M, N, P, Q \) and \( R \).

If possible, let us assume that \( \mu \) is another fixed point of \( L, M, N, P, Q \) and \( R \).

\[ \therefore \xi \left\{ \int_0^\infty f(t) \, dt \right\} = \xi \left\{ \int_0^\infty f(t) \, dt \right\} - \eta \left\{ \int_0^\infty f(t) \, dt \right\} \]

i.e. \[ \xi \left\{ \int_0^\infty f(t) \, dt \right\} < \xi \left\{ \int_0^\infty f(t) \, dt \right\}, \text{ which is again a contradiction.} \]

Hence finally we will have \( u = \mu \).

Thus \( u \) is the unique common fixed point of \( L, M, N, P, Q \) and \( R \).

Corollary 2.2: Let \((X, G)\) be a complete G-metric space and \( L, M, N, P : X \to X \) be mappings such that

i) \( \xi \left\{ \int_0^\infty f(t) \, dt \right\} < \xi \left\{ \int_0^\infty f(t) \, dt \right\}, \text{ which is a contradiction.} \therefore \text{ we get } Mu = u \]

ii) \( \xi \left\{ \int_0^\infty f(t) \, dt \right\} \leq \xi \left\{ \int_0^\infty f(t) \, dt \right\} - \eta \left\{ \int_0^\infty f(t) \, dt \right\} \]

for all \( x, y, z \in X \) where \( \xi : [0, \infty) \to [0, \infty) \) is a continuous and non-decreasing function , \( \eta : [0, \infty) \to [0, \infty) \) is a lower semi continuous and non-decreasing function such that \( \xi(t) = \eta(t) = 0 \) if and only if \( t = 0 \), also \( f : [0, \infty) \to [0, \infty) \) is a Lebesgue integrable function which is summable on each compact subset of \( R^+ \), non negative and such that for each \( \varepsilon > 0 \),

\[ \int_0^\varepsilon f(t) \, dt > 0 \]

iii) The pairs \((L, P)\), \((M, P)\), \((N, P)\) are weakly compatible.

Then \( L, M, N, P \) have a unique common fixed point in \( X \).

Proof: By taking \( P = Q = R \) in Theorem 2.1 we get the proof.

Corollary 2.3: Let \((X, G)\) be a complete G-metric space and \( L, P : X \to X \) be mappings such that
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\[ L(X) \subseteq P(X) \]

\[ \xi(t) = \eta(t) = 0 \text{ if and only if } t = 0, \]

\[ \int_{0}^{\xi} f(t) \, dt > 0 \]

iii) The pair \((L, P)\) is weakly compatible.

\[ \text{Then } L, P \text{ have a unique common fixed point in } X. \]

**Proof:** By substituting \( L = M = N \) and \( P = Q = R \) in Theorem 2.1 we get the proof.

**Remark:** The Corollary 2.3 is the result proved by Vishal Gupta and Naveen Mani [5] in complete metric space.

### IV. References


