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Tolerance Interval for Exponentiated Exponential Distribution Based on Grouped Data

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Abstract:- Upper β-expectation and β-content γ-level tolerance interval for exponentiated exponential distribution based on grouped data are considered. The adequacy of proposed tolerance intervals are evaluated using simulation study and through real life data due to Lawless (1982).

Keywords : Exponentiated Exponential Distribution, Upper β expectation and β content γ level tolerance interval, Maximum likelihood estimator, grouped data.

I. INTRODUCTION

Gupta et al. (1998) introduced the Exponentiated Exponential Distribution (EE) as a generalization of the standard exponential distribution. Gupta and Kundu (2001) Studied this distribution and observed that it can be used quite effectively in analyzing many life time data particularly in place of two parameter gamma and Weibull distributions. In their series of papers on EE distribution (also named as Generalized exponential distribution), they discussed different estimation procedures, testing of hypothesis, construction of confidence interval, estimating the stress and strength parameter and closeness of generalized exponential with Weibull, gamma and lognormal distribution. Shirke et al. (2005) obtained β-expectation and β-content γ-level tolerance interval for this distribution based on ungrouped data. Many times in a life testing problem, it is not possible to record exact time of failure of a components due to several reasons. Hence it is more economical to observe no of failures of components in predefined time intervals which form grouped data. The main aim of this paper is to construct β expectation and β content γ level tolerance interval for EE distribution based on grouped data.

Let U be a statistic based on data observed from a distribution with density function f(x, $\underline{\theta}$) where $\underline{\theta}$ represents a vector of unknown parameters then the interval (-∞, U) is a β-expectation tolerance interval (TI) if

$$E\left[\int_{-\infty}^{U} f(x,\theta)dx\right] = \beta \quad \text{for every } \underline{\theta} \in \Theta \qquad (1.1)$$

and interval (-∞, U) is upper β content γ level TI if

$$P\left[\int_{-\infty}^{U} f(x,\theta)dx \geq \beta\right] = \gamma \quad \text{for given } \beta, \gamma \in (0,1). \qquad (1.2)$$

C is called the coverage of TI (-∞, U) if $C = \int_{-\infty}^{U} f(x,\theta)dx$.

One sided TI's when f(x,θ) is a two parameter EE distribution based on maximum likelihood estimators for ungrouped data were discussed by Shirke et al. (2005). This paper extends the procedure for setting TI based on grouped data. In section 2 we provide upper β-expectation TI along with its approximate expected coverage. β-content γ-level TI based on MLE's of θ are discussed in section 3. We study the performance of both these TI using simulation technique in section 4. In section 5 we illustrate the practical applications of the procedure by applying it to real life data set.

II. β-EXPECTATION TOLERANCE INTERVAL

Gupta and Kundu (2001) defined EE distribution in the following way. Let Y be a two parameter EE random variable with distribution function

$$F(y,\lambda,\alpha)=(1-e^{-\lambda y})^{\alpha} \quad \alpha,\lambda,y>0, \qquad (2.1)$$

Therefore corresponding probability density function is given by

$$f(y,\lambda,\alpha)=\alpha\lambda(1-e^{-\lambda y})^{\alpha-1}e^{-\lambda y}. \quad ,y>0 \qquad (2.2)$$

Suppose X_j , j=1,2,….,k be the number of observations in the interval $(t_{j-1}, t_j]$ with $t_0=0$ and $t_k=\infty$ such that $\sum_{i=1}^{k} x_i = n$. Thus we have grouped data from the underlying distribution .

The log likelihood function for the grouped data can be written as

$L(\lambda,\alpha/\underline{x}) = C+\sum_{i=1}^{k} x_i \ln\left[F(t_i)-F(t_{i-1})\right]$, where c is independent of λ and α and F(.) is defined in (2.1).

Therefore to obtain MLE of λ and α, we differentiate $L(\lambda,\alpha)$ partially w.r.t. λ and α and equating it to zero. This gives

$$\frac{\partial L}{\partial\alpha}=\sum_{1=1}^{k}\frac{x_i\ln(1-e^{-\lambda t_i})}{1-\left(\dfrac{1-e^{-\lambda t_{i-1}}}{1-e^{-\lambda t_i}}\right)^{\alpha}}-\sum_{i=1}^{k}\frac{x_i\ln(1-e^{-\lambda t_{i-1}})}{\left(\dfrac{1-e^{-\lambda t_i}}{1-e^{-\lambda t_{i-1}}}\right)^{\alpha}-1}=0$$

$$\frac{\partial L}{\partial\lambda}=\sum_{1=1}^{k}\frac{\alpha t_i x_i e^{-\lambda t_i}(1-e^{-\lambda t_i})^{\alpha-1}}{(1-e^{-\lambda t_i})^{\alpha}-(1-e^{-\lambda t_{i-1}})^{\alpha}}-\sum_{i=1}^{k}\frac{\alpha t_{i-1}x_i e^{-\lambda t_{i-1}}(1-e^{-\lambda t_{i-1}})^{\alpha-1}}{(1-e^{-\lambda t_i})^{\alpha}-(1-e^{-\lambda t_{i-1}})^{\alpha}}=0$$

It is clear that no closed form solution is possible. We can use Newton-Raphson method to solve above two equations.

The β^{th} percentile of (2.2) is given by $X_\beta(\underline{\theta})=\lambda^{-1}\ln(1-\beta^{1/\alpha})$. Since $\underline{\theta}=(\lambda,\alpha)$ is unknown, we replace it by its MLE $\hat{\underline{\theta}}=(\hat{\lambda},\ \hat{\alpha})$ and we propose approximate upper β expectation TI as $I_1(\underline{x})=\left(0,\hat{\lambda}^{-1}\ln(1-\beta^{1/\hat{\alpha}})\right)$.

Expected coverage of $I_1(\underline{X})$ is given by following theorem.

Theorem 1: An expected coverage of $I_1(\underline{X})$ is given by

$$E[\ F\{X_\beta(\hat{\underline{\theta}});\ \underline{\theta}\}] = \quad \beta+A(\underline{\theta})\sigma_1^2+B(\underline{\theta})\sigma_2^2+C(\underline{\theta})\sigma_{12}, \qquad (2.3)$$

where $A(\underline{\theta})=0.5\alpha(1-\alpha)[G(t)]^{\alpha-2}[G'_\lambda(t)]^2+\alpha[G(t)]^{\alpha-1}\left[\dfrac{G''_{x\lambda}(t)G'_\lambda(t)}{G'_x(t)}-0.5G''_{\lambda\lambda}(t)\right]$,

$B(\underline{\theta})=\alpha^{-1}[G(t)]^{\alpha}\log G(t)\,[1+0.5\alpha\log G(t)]$, $C(\underline{\theta})=[G(t)]^{\alpha}\log G(t)\left[\dfrac{(\alpha-1)G'_\lambda(t)}{G(t)}+\dfrac{G''_{x\lambda}(t)}{G'_x(t)}\right]$.

$G(t)=\left(1-e^{-\lambda t}\right)$, $G'_x(t)=\partial G(t)/\partial x$, $G'_\lambda(t)=\partial G(t)/\partial\lambda$, $G''_{x\lambda}(t)=\partial^2 G(t)/\partial x\partial\lambda$,

$G''_{\lambda\lambda}(t) = \partial^2 G(t)/\ \partial\lambda^2$ while σ_1^2, σ_2^2 are asymptotic variances of $\hat{\lambda}$ and $\hat{\alpha}$ respectively and σ_{12} = Cov($\hat{\lambda}$, $\hat{\alpha}$) is an asymptotic covariance of $\hat{\lambda}$, $\hat{\alpha}$.

Proof : Follows from the result of Atwood (1984). We omit the same for brevity.

The performance of $I_1(\underline{X})$ is studied using simulation experiments and has been reported in the Section 4. In the following we obtain β-content γ-level TI for the distribution (2.1).

III. β-CONTENT γ-LEVEL TOLERANCE INTERVAL

Let $I_2(\underline{X})$ = (0, $\delta \hat{X}_\beta$) be an upper β-content γ-level TI for the distribution having cdf (2.1). The factor δ > 0 is to be determined such that $I_2(\underline{X})$ is a β-content γ-level TI for β∈(0, 1) and γ∈(0, 1).

i.e. $P\left[F\left(\delta\,\hat{X}_\beta;\lambda,\alpha\right) \ge \beta\right] = \gamma$. This implies $P\left[\hat{X}_\beta \ge \dfrac{\ln\left(1-\beta^{1/\alpha}\right)}{-\lambda\delta}\right] = \gamma$ (3.1)

Assuming consistency and asymptotic normality of the MLE's $\hat{\lambda}$ and $\hat{\alpha}$, we have $\hat{X}_\beta(\underline{\theta})$ is asymptotically normal with mean $X_\beta(\underline{\theta})$ and variance $\dfrac{\sigma^2(\underline{\theta})}{n}$, where $\sigma^2(\underline{\theta}) = H\Sigma H'$ with Σ as a variance covariance matrix of $\hat{\underline{\theta}} = (\hat{\lambda},\ \hat{\alpha})$ and $H = \left[\dfrac{\partial x_\beta(\underline{\theta})}{\partial\lambda},\quad \dfrac{\partial x_\beta(\underline{\theta})}{\partial\alpha}\right]$.

If Z is a standard normal variate then we can write from (3.1)

$$P\left[Z \le \left\{\frac{\ln(1-\beta^{1/\alpha})}{\lambda\delta} - x_\beta(\underline{\theta})\right\}\left(\frac{\sqrt{n}}{\sigma(\underline{\theta})}\right)\right] = 1-\gamma,$$

Suppose $Z_{1-\gamma} = \left\{\dfrac{\ln(1-\beta^{1/\alpha})}{\lambda\delta} - x_\beta(\underline{\theta})\right\}\left(\dfrac{\sqrt{n}}{\sigma(\underline{\theta})}\right)$, where $Z_{1-\gamma}$ is the $100(1-\gamma)^{th}$ percentile of the standard normal variate then we have $\delta = \left[1 - \dfrac{\lambda\, Z_{1-\gamma}\sigma(\underline{\theta})}{\ln(1-\beta^{1/\alpha})\sqrt{n}}\right]^{-1}$.

Note that δ depends on both the parameters λ and α. Replacing λ and α in δ by their respective MLEs an asymptotic upper β-content γ-level TI $I_2(\underline{X})$ is

$$\left(0\ ,\ -\left[1+\frac{\hat{\lambda}\, Z_{1-\gamma}\sigma(\hat{\underline{\theta}})}{\ln\left(1-\hat{\beta}^{1/\hat{\alpha}}\right)\sqrt{n}}\right]^{-1}\frac{\ln\left(1-\beta^{1/\hat{\alpha}}\right)}{\hat{\lambda}}\right) \qquad (3.2)$$

The performance of $I_2(\underline{X})$ is studied using simulation experiments and has been reported in the Section 5.

IV. SIMULATION STUDY

The approximate value of the actual coverage of the proposed β expectation TI is given by (2.3). The performance of the said TI is illustrated by simulation technique by replacing λ and α by its MLE's $\hat{\lambda}$ and $\hat{\alpha}$. In the simulation study of $I_1(\underline{X})$, we generate n (=10, 25, 50, 75, 100) observations from EE distribution with λ=2 , α=2 and groupe into five equal spaced intervals as 0- 0.5, 0.5- 1.0, 1.0- 1.5, 1.5- 2.0 and above 2.0. MLE's of λ and α are otained and are used to compute upper β- expectation TI of $I_1(\underline{X})$ for β=.90, .95, .975 and .99. Repeating the above procedure 10,000 times and estimate expected coverage reported in Table 4.1. Same procedure is repeated by grouping observations into five unequal spaced intervals as 0- 0.4, 0.4- 0.9, 0.9- 1.5, 1.5- 2.2 and above 2.2. The expected coverage reported in Table 4.2. We observe from Table 4.1 and 4.2 that for small sample size TI underestimates expected coverage while as sample size increases coverage converges to the desired level. It verifies the consistency property of the proposed interval.

Table 4.1: Expected coverage of β- expectation TI (equally spaced)

β	n				
	10	25	50	75	100
0.90	.8518	.8794	.8854	.8934	.8982
0.95	.9042	.9422	.9456	.9470	.9482
0.975	.9330	.9650	.9687	.9728	.9766
0.99	.9520	.9802	.9842	.9864	.9880

Table 4.2: Expected coverage of β- expectation TI (unequally spaced)

β	n				
	10	25	50	75	100
0.90	.8594	.8874	.8923	.8970	.8990
0.95	.9146	.9328	.9384	.9428	.9500
0.975	.9386	.9660	.9821	.9740	.9762
0.99	.9596	.9826	.9856	.9890	.9900

Upper β-content γ-level TI given in (3.2) require asymptotic variance covariance matrix of the MLE's $\hat{\lambda}$ and $\hat{\alpha}$ which involve complicated integrals to be solved numerically. This problem is resolved by using bootstrap technique. In the following section we illustrate an application of the procedure by applying it to real life data set.

V. APPLICATION

Lawless data set (1982, page 228) is best fitted to EE distribution than two parameter Weibull and Gamma as reported by Gupta and Kundu (2001).The data are regarding arose in test on endurance of deep groove ball bearings. The data are the class of number of million resolutions before failure for 23 ball bearings in the life test with equally spaced intervals as

Class interval : 0-35 35-70 70-105 105-140 140-
No of ball bearings : 3 12 3 4 1

For equally spaced data MLE's of λ and α are 0.0326 and 4.9536 respectively and upper β expectation tolerance limits (say $U_1(\underline{X})$) for various β's are

β : 0.90 0.95 0.975 0.99
$U_1(\underline{X})$: 118.4462 140.3613 161.9405 190.2356

If the above data are with unequally spaced intervals as,

Class interval : 0-35 35-55 55-80 80-100 100-
No of ball bearings : 3 7 5 3 5

MLE's of λ and α are 0.0302 and 4.4747 respectively and upper β expectation tolerance limits (say $U_2(\underline{X})$) for various β's are

β : 0.90 0.95 0.975 0.99
$U_2(\underline{X})$: 124.6725 148.3370 171.6484 202.2216

Using bootstrap technique we generate 5000 random samples (with replacement) each of size 23 from the original data and group them with intervals equally spaced . For each of the bootstrap sample we compute MLE's of λ and α. Based on such 5000 MLE's we obtain variance covariance matrix of MLE's and is used to propose asymptotic upper β content γ level tolerance limit (TL) with combinations of β=.90, .95, .975,.99 and γ= .90, .95 and tabulated in Table 5.1. We also generate 5000 random bootstrap samples (with replacement) each of size 23 from the original data and group them with intervals unequally spaced. By repeating the same procedure as above ,we propose asymptotic upper β-content γ-level tolerance limit and reported in Table 5.2.

Table 5.1: Upper β-Content γ-level TL (equally spaced)

β	$\sigma(\underline{\theta})$	γ=.90		γ=.95	
		δ	$U(\underline{X})$	δ	$U(\underline{X})$

0.90	20.0795	1.0475	124.0667	1.0615	125.7353
0.95	23.5777	1.0470	146.9582	1.0609	148.9154
0.975	27.8883	1.0482	169.7526	1.0626	172.0740
0.99	34.3264	1.0507	199.8735	1.0658	202.7460

Table 5.2: Upper β-Content γ-level TL (unequally spaced)

β	σ($\underline{\theta}$)	γ=.90		γ=.95	
		δ	U($\underline{X}$)	δ	U($\underline{X}$)
0.90	20.9688	1.0471	130.5397	1.0610	132.2806
0.95	26.4637	1.0501	155.7630	1.0650	157.9746
0.975	32.4806	1.0533	180.7905	1.0692	183.5240
0.99	40.8536	1.0571	213.7621	1.0742	217.2288

It is observed that β-content γ-level TI for equally spaced intervals has a wider length as against the unequally spaced intervals.

The simulation study indicates that for small sample size TI underestimates expected coverage while as sample size increases coverage converges to desired level. The performance of both the proposed TI's are satisfactory and can be used in practice. The method developed here can be extended for distributions belongs to exponentiated scale family of distributions such as exponentiated Weibull, Exponentiated gamma or Exponentiated Rayleigh distribution.

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