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Convexity, Concavity, and Inflection Behavior of Legendre Polynomials of Higher Degree

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Abstract
This study investigates the convexity and concavity properties of higher-order Legendre polynomials through analytical and numerical approaches. Closed-form expressions of the second derivatives of the Legendre polynomials were utilized to determine the inflection points within the standard interval $[-1,1]$*. Special emphasis is given to the seventh- and eighth-degree Legendre polynomials. Theoretical inflection points were obtained and verified numerically using finite-difference approximations in the R programming environment. Convex and concave regions were visually distinguished by color-coded plots. The results demonstrate that higher-degree Legendre polynomials exhibit increasingly complex oscillatory curvature structures characterized by alternating convex and concave intervals. These findings provide a detailed geometric interpretation of the curvature behavior of orthogonal polynomials and offer a useful framework for their application in approximation theory, spectral methods, and regression modeling.*
Keywords: *Legendre polynomials, convexity, concavity. inflection points. second derivative.*

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I. Introduction

Legendre polynomials constitute one of the most fundamental families of orthogonal polynomials in mathematical analysis and applied sciences. Defined on the interval $[-1,1]$and orthogonal with respect to the unit weight function, these polynomials form a natural basis for a wide range of theoretical and computational applications. They play a central role in approximation theory, spectral methods for differential equations, numerical integration, and statistical modeling, particularly in polynomial regression and orthogonal expansions (Burden and Faires, 2011).

The geometric properties of basis functions, such as convexity and concavity, are of crucial importance in both theoretical investigations and practical applications. In regression analysis, for instance, the curvature structure of basis functions directly affects the flexibility, stability, and interpretability of fitted models. Similarly, in numerical analysis and spectral approximations, curvature behavior is closely related to oscillation patterns and error propagation. Despite the extensive literature on Legendre polynomials, most studies primarily focus on their orthogonality, recurrence relations, and approximation properties, while their detailed curvature structures have received comparatively less attention (Rockafellar, 1970).

Convexity and concavity are classical concepts in mathematical analysis and are characterized by the sign of the second derivative. The points at which the second derivative vanishes define inflection points, separating convex and concave regions. For higher-degree polynomials, the number and distribution of such inflection points carry important geometric and numerical implications. In particular, the alternating convex–concave structure of higher-order orthogonal polynomials reflects their oscillatory nature and directly influences their performance in approximation and regression-based modeling (Boyd, 2001; Mason &Handscomb, 2003).

In recent years, the increasing use of orthogonal polynomial bases in data-driven and statistical learning frameworks has further highlighted the need for a precise understanding of their local curvature behavior. Multicollinearity reduction, numerical stability, and adaptive basis construction in regression and spectral modeling are all closely linked to the geometric characteristics of the chosen basis functions. However, systematic curvature-based analyses of higher-order Legendre polynomials supported by both analytical derivations and numerical verification remain relatively scarce in the literature(Szegö, 1939; Shen, Tang, & Wang, 2011).

Motivated by this gap, the present study provides a comprehensive convexity–concavity analysis of Legendre polynomials based on second-derivative characterization and numerical validation. Special emphasis is placed on the seventh- and eighth-degree Legendre polynomials, which exhibit rich curvature structures representative of higher-order behavior. Analytical inflection points are derived from symbolic second derivatives and verified numerically using finite-difference approximations implemented in the R programming environment. Convex and concave regions are illustrated through color-coded graphical representations (Tufte, 2001).

The main contributions of this study are threefold. First, it provides a systematic mathematical characterization of the inflection-point structure of higher-order Legendre polynomials. Second, it introduces a practical and reproducible numerical framework in R for curvature-based analysis and visualization. Third, it establishes a direct link between the geometric properties of Legendre polynomials and their role in statistical regression and spectral approximation methods.

II. Materials And Methods

Legendre polynomials

The first kind of Legendre functions reduce to polynomials when n is zero or a positive even integer, whereas the second kind corresponds to positive odd integer values of n. The constants a_0 and a_1 are arbitrary and are chosen such that each polynomial satisfies the normalization condition P□ (1) = 1 (Altın, 2011).

The first property of the Legendre polynomials is the Rodrigues formula (Lima, 2025).

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \quad n \in N_0$$

The Rodrigues formula indicates that P(X) is an nth degree polynomial. Furthermore, for n odd, the polynomial is a n odd function, whereas for n even, it is an even function.

Legendre's Functions of First and Second Kinds

$P_n(x)$ represents Legendre's function of the first type, also known as Legendre's polynomial of degree n (Indian Institute of Technology Guwahati, n.d., 2025).

$$P_n(x) = \frac{1.3.5 \ldots .(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} - \cdots\right]$$

It can be also written $P_n(x)$ in a compact from as:

$$P_n(x) = \sum_{r=0}^{[n/2]}(-1)^r \frac{(2n-2r)!}{2^n r!\,(n-2r)!\,(n-r)}x^{n-2r}$$

$$\text{where} \quad [n/2] = \begin{cases} \frac{n}{2}, \text{if n is even} \\ (n-1)/2, \quad \text{if n is odd} \end{cases}$$

Legendre's function of the second kind is denoted by Qn(x) and is defined by

$$Q_n(x) = \frac{n!}{1.3.5 \ldots .(2n+1)}\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)}x^{-n-5} - \cdots\right]$$

Some Legendre polynomials are given below (Altın, 2011).

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2-1)$$
$$P_3(x) = \frac{1}{2}(5x^3-3x)$$
$$P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$$
$$P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$$

The sixth and seventh Legendre polynomials are as follows (Badmus and Subair, 2024).

$$P_6(x) = \frac{1}{16}(231x^6-315x^4+105x^2-5)$$
$$P_7(x) = \frac{1}{16}(429x^7-693x^5+315x^3-35x)$$

Convex Function

A convex function is defined as follows.

Let S be a non-empty set defined in En. If f(x) is a convex function on S, then

$$f(\lambda x_2 + (1-\lambda)x_1) \le \lambda f(x_2) + (1-\lambda)f(x_1)$$

Here, $x_1, x_2 \in S$ and $\lambda \in [0,1]$.

If f(x) is a strictly convex function, then

$$f(\lambda x_2 + (1-\lambda)x_1) < \lambda f(x_2) + (1-\lambda)f(x_1)$$

(Apaydın, 1996). In this case, the function$-$f(x) will be concave. For example, the convexity of the function$f(x) = x^2, x \in \mathbb{R}$is shown below. The function f(x) is denoted as

$$f(\lambda x_2 + (1-\lambda)x_1) \le \lambda f(x_2) + (1-\lambda)f(x_1), \quad 0 \le \lambda \le 1.$$

$$(\lambda x_2 + (1-\lambda)x_1)^2 \le \lambda (x_2)^2 + (1-\lambda)(x_1)^2$$
$$\lambda^2 x_2^2 + 2\lambda(1-\lambda)x_1x_2 + (1-\lambda)^2 x_1^2 \le \lambda x_2^2 + (1-\lambda)x_1^2$$
$$\lambda^2 x_2^2 + 2\lambda(1-\lambda)x_1x_2 + (1-\lambda)^2 x_1^2 - \lambda x_2^2 - (1-\lambda)x_1^2 \le 0$$
$$-\lambda(1-\lambda)x_2^2 + 2\lambda(1-\lambda)x_1x_2 - \lambda(1-\lambda)x_1^2 \le 0$$
$$-\lambda(1-\lambda)(x_2^2 - 2x_1x_2 + x_1^2) \le 0$$
$$-\lambda(1-\lambda)(x_1 - x_2)^2 \le 0$$

Since the function$f(x) = x^2$is convex.A similar example of a concave function can be given. The concave nature of the function$f(x) = \sqrt{x},\ x \in \mathbb{R}$is shownbelow.

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda)f(x_2), \qquad 0 \le \lambda \le 1$$
$$\sqrt{\lambda x_1 + (1-\lambda)x_2} \le \lambda\sqrt{x_1} + (1-\lambda)\sqrt{x_2}$$
$$\lambda x_1 + (1-\lambda)x_2 \le \lambda^2 x_1 + (1-\lambda)^2 x_2 + 2\lambda(1-\lambda)\sqrt{x_1x_2}$$
$$\lambda x_1 + x_2 - \lambda x_2 \le \lambda^2 x_1 + (1 - 2\lambda + \lambda^2)x_2 + 2\lambda(1-\lambda)\sqrt{x_1x_2}$$
$$\lambda x_1 + x_2 - \lambda x_2 \le \lambda^2 x_1 + x_2 + \lambda^2 x_2 - 2\lambda x_2 + (2\lambda - 2\lambda^2)\sqrt{x_1x_2}$$
$$\lambda x_1 \le \lambda^2 x_1 + \lambda^2 x_2 - \lambda x_2 + 2\lambda\sqrt{x_1x_2} - 2\lambda^2\sqrt{x_1x_2}$$

For example, let $\lambda = 0.6$.

$$0.6x_1 \le 0.36x_1 + 0.36x_2 - 0.6x_2 + 1.2\sqrt{x_1x_2} - 0.72\sqrt{x_1x_2}$$
$$0.6x_1 \le 0.36x_1 - 0.24x_2 + 0.48\sqrt{x_1x_2}$$
$$0 \le -0.24x_1 - 0.24x_2 + 0.48\sqrt{x_1x_2}$$
$$0 \nleq -0.24(x_1 - x_2 + 2\sqrt{x_1x_2})$$

Therefore, this inequality is not satisfied and the function $f(x) = \sqrt{x}$ is concave.
The second derivative test is a powerful tool for identifying convexity (Niculescu and Persson, 2006).
Given a function $y = f(x)$, if $f''(x) \ge 0$ on an interval (a, b), then f(x) is convex on that interval. If $f''(x) < 0$ on an interval (a, b), then f is concave on that interval.
Given a function $y = f(x)$, the sign of its second derivative determines its curvature. If on an interval (a, b),

$$f''(x) \ge 0,$$

then the function $f$is convex on this interval. Conversely, if on the same interval

$$f''(x) < 0,$$

then the function $f$is concave on this interval (Stewart, 2016).

III. Results

The convexity condition of the first Legendre polynomials is summarized below.

$$P_0(x) = 1$$
$$P_0''(x) = 0$$

It is neither convex nor concave. It is a smooth function.

$$P_1(x) = x$$
$$P_1''(x) = 0$$

Straight line again.

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$
$$P_2'(x) = 3x$$
$$P_2''(x) = 3 > 0$$

It is convex in all $\mathbb{R}$.

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$
$$P_3'(x) = \frac{1}{2}(15x^2 - 3)$$
$$P_3''(x) = 15x$$

Convex intervals are as follows: If $x > 0$, it is convex, if $x < 0$, it is concave, and if $x = 0$, it is a curvature change point.

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_4'(x) = \frac{1}{8}(140x^3 - 60x)$$
$$P_4''(x) = \frac{1}{8}(420x^2 - 60)$$

Zero points:

$$420x^2 - 60 = 0 \Rightarrow x^2 = \frac{1}{7} \Rightarrow x = \pm\frac{1}{\sqrt{7}} \approx \pm 0.378$$

According to the results obtained using the numerical second derivative, the Legendre polynomial $P_4(x)$ exhibits a convex structure in the range $\mid x \mid > 1/\sqrt{7}$ and a concave structure in the range $\mid x \mid < 1/\sqrt{7}$. The curvature change points are also verified numerically.

$$P_5(x) = \frac{x(63x^4 - 70x^2 + 15)}{8}$$

The second derivative of $P_5(x)$is

$$P_5''(x) = \frac{105}{2}\, x(3x^2 - 1).$$

Inflection points of $P_5(x)$
($P_5''(x) = 0$, $[-1,1]$):

$$x = -0.577350,\ 0.0,\ 0.577350(\pm 1/\sqrt{3}, 0)$$

Convex/Concave intervals (as open intervals):

- (-1,-0.577350): concave
- (-0.577350,0): convex
- (0,0.577350): concave
- (0.577350,1): convex

$P_5(x)$ shows a symmetric structure; since its second derivative has a factor of x and$(3x^2 - 1)$, there are three inflection points and the convex/concave regions change regularly according to the range.

$$P_6(x) = \frac{231x^6 - 315x^4 + 105x^2 - 5}{16}$$

Second derivative of $P_6(x)$

$$P_6''(x) = \frac{105}{8}\,(33x^4 - 18x^2 + 1)$$

Skew change (real roots, [-1,1]):

$$x \approx -0.694747,\ -0.250563,\ 0.250563,\ 0.694747$$

Convex/Concave ranges:

- $(-1, -0.694747)$: Convex
- $(-0.694747, -0.250563)$: Concave
- $(-0.250563, 0.250563)$: Convex
- $(0.250563, 0.694747)$: Concave
- $(0.694747, 1)$: Convex

For P_6, P_6''contains four real roots, which shows that the polynomial has successive convex/concave transitions in [-1,1].

$$P_7(x) = \frac{x(429x^6 - 693x^4 + 315x^2 - 35)}{16}$$

Second derivative of $P_7(x)$

$$P_7''(x) = \frac{63}{8}\, x(143x^4 - 110x^2 + 15)$$

Skew change (roots, [-1,1]):

$$x \approx -0.769455,\ -0.420915,\ 0.0,\ 0.420915,\ 0.769455$$

Convex/Concave ranges:

- $(-1, -0.769455)$: Concave
- $(-0.769455, -0.420915)$: Convex
- $(-0.420915, 0)$: Concave
- $(0, 0.420915)$: Convex
- $(0.420915, 0.769455)$: Concave
- $(0.769455, 1)$: Convex

In odd polynomials, $x = 0$is necessarily an inflection point due to the x factor; P_7 shows this and ±two symmetric inflections.
The graph showing the intervals in which $P_7(x)$is convex and concave is presented in Figure 1.

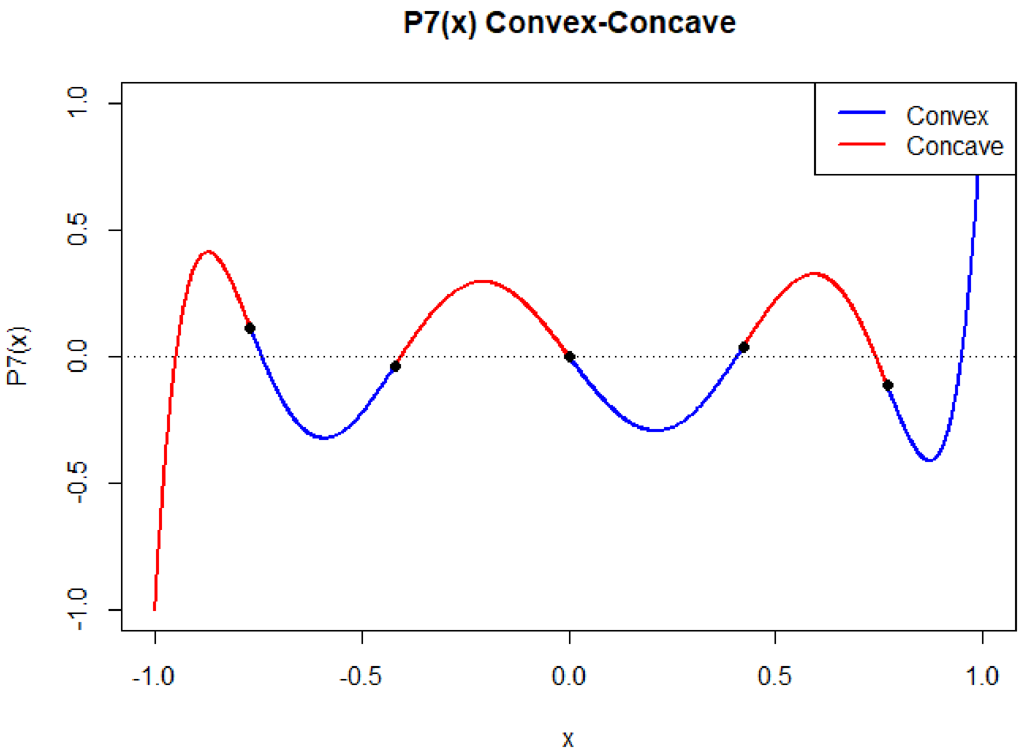

Figure 1. Convex and Concave ranges in $P_7(x)$

$$P_8(x) = \frac{6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35}{128}$$

Second derivative of $P_8(x)$

$$P_8''(x) = \frac{315}{16}(143x^6 - 143x^4 + 33x^2 - 1)$$

Skew change (roots, [-1,1]):

$$x \approx -0.855402, \; -0.627378, \; -0.331630, 0.0, 0.331630, 0.627378, 0.855402$$

Convex/Concave ranges:
Intervals alternate between roots; in example order (left to right):

- $(-1, -0.855402)$: Convex
- $(-0.855402, -0.627378)$: Concave
- $(-0.627378, -0.331630)$: Convex
- $(-0.331630, 0)$: Concave
- $(0, 0.331630)$: Convex
- $(0.331630, 0.627378)$: Concave
- $(0.627378, 0.855402)$: Convex
- $(0.855402, 1)$: Concave

For P_8, P_8''contains a low degree polynomial and there are 7 inflection points in [-1,1]; due to symmetry the points are ± conjugate and $x = 0$is again an inflection point.
The graph showing the intervals in which $P_8(x)$is convex and concave is presented in Figure 2.

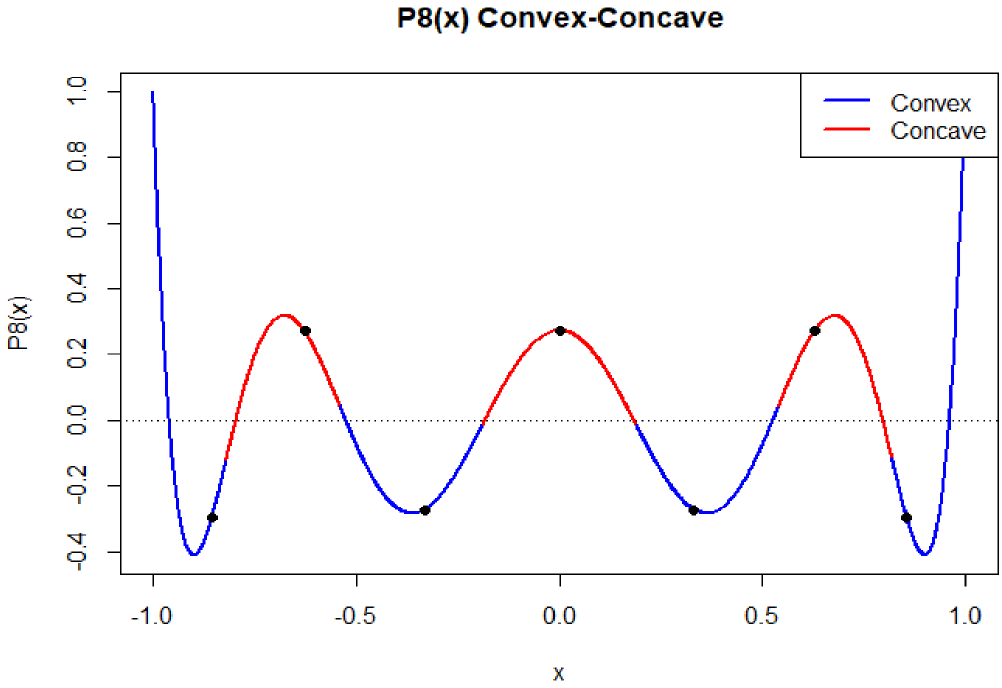

Figure 2. Convex and Concave ranges in $P_8(x)$

$$P_9(x) = \frac{x(12155x^8 - 25740x^6 + 18018x^4 - 4620x^2 + 315)}{128}$$

Second derivative of $P_9(x)$

$$P_9''(x) = \frac{495}{16}\,x\,(221x^6 - 273x^4 + 91x^2 - 7)$$

Skew change (roots, [-1,1]):

$x \approx -0.881409,\ -0.692061,\ -0.441433,\ -0.151632, 0.151632, 0.441433, 0.692061, 0.881409$

Since $P_9$is odd-degree, its second derivative contains a factor of x; this polynomial exhibits high-degree varying curvature behavior and has 8 inflection points in [-1,1].

$$\mathrm{P}_{10}(\mathrm{x}) = \frac{46189\mathrm{x}^{10} - 109395\mathrm{x}^8 + 45045\mathrm{x}^6 - 15015\mathrm{x}^4 + 3465\mathrm{x}^2 - 63}{256}$$

Second derivative of $\mathrm{P}_{10}(\mathrm{x})$

$$\mathrm{P}_{10}''(\mathrm{x}) = \frac{495}{128}\,(4199\mathrm{x}^8 - 6188\mathrm{x}^6 + 2730\mathrm{x}^4 - 364\mathrm{x}^2 + 7)$$

Skew change (roots, [-1,1]):

$\mathrm{x} \approx -0.881409,\ -0.692061,\ -0.441433,\ -0.151632, 0.151632, 0.441433, 0.692061, 0.881409$

Convex/Concave ranges:

- $(-1, -0.881409)$: Convex
- $(-0.881409, -0.692061)$: Concave
- $(-0.692061, -0.441433)$: Convex
- $(-0.441433, -0.151632)$: Concave
- $(-0.151632, 0.151632)$: Convex
- $(0.151632, 0.441433)$: Concave
- $(0.441433, 0.692061)$: Convex
- $(0.692061, 0.881409)$: Concave
- $(0.881409, 1)$: Convex

$\mathrm{P}_{10}$is a higher order polynomial of even degree; due to the polynomial nature of the second derivative, multiple (8) inflection points occur in [-1,1] and convex/concave regions alternate.

IV. Conclusion

This study has systematically examined the convexity and concavity behavior of Legendre polynomials through second-derivative analysis and numerical verification. The findings demonstrate that, except for low-degree cases, Legendre polynomials exhibit multiple alternating convex and concave regions within the interval $[-1,1]$. The number of inflection points increases with the polynomial degree, leading to progressively more complex curvature structures.

The detailed numerical analysis of $P_7(x)$and $P_8(x)$confirmed the existence of multiple symmetric inflection points and provided a clear visualization of curvature transitions. The computational framework implemented in the R environment successfully validated the analytical results and offered an efficient tool for curvature-based analysis of orthogonal polynomials.

These results underline the importance of curvature structure when Legendre polynomials are employed in practical applications such as spectral approximation, regression modeling, and numerical solution of differential equations. The pronounced convex–concave alternations observed in higher-order polynomials highlight their strong flexibility in modeling nonlinear phenomena. Future research may extend this approach to multivariate Legendre expansions and curvature analysis in high-dimensional approximation and machine learning models.

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